Symmetry Reductions and Exact Solutions of the Two-Layer Model in Atmosphere

Zhongzhou Dong[a,d], Fei Huang[b,c], and Yong Chen[a,c]

[a] Shanghai Key Laboratory of Trustworthy Computing, East China Normal University, Shanghai, 200062, China
[b] Physical Oceanography Laboratory and Ocean-Atmosphere Interaction and Climate Laboratory, Ocean University of China, Qingdao, 266100, China
[c] Nonlinear Science Center and Department of Mathematics, Ningbo University, Ningbo, 315211, China
[d] College of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo, 454000, China

Reprint requests to Y. C.; E-mail: ychen@sei.ecnu.edu.cn

Z. Naturforsch. **66a,** 75 – 86 (2011); received March 3, 2010 / revised June 24, 2010

By means of the classical symmetry method, we investigate the two-layer model in atmosphere. The symmetry group of two-layer model equations is studied and its corresponding group invariant solutions are constructed. Ignoring the discussion of the infinite-dimensional subalgebra, we construct the optimal system of one-dimensional and two-dimensional group invariant solutions. Furthermore, using the associated vector fields of the obtained symmetry, we give out the reductions by one-dimensional and two-dimensional subalgebras, and some explicit solutions of two-layer model equations are obtained. For some interesting solutions, the figures are given out to show their properties. Some solutions can describe the horizontal structure of tropical cyclones (TC). Especially, a new solution of double-eyewall structure of TCs is firstly found in this two-layer model.

Key words: Two-Layer Model Equations; Classical Lie Symmetry Method; Optimal System; Explicit Solution.

1. Introduction

Symmetry group techniques provide one method for obtaining exact solutions of partial differential equations [1 – 4]. Since Sophus Lie [1] set up the theory of the Lie point symmetry group, the standard method had been widely used to find Lie point symmetry algebras and groups for almost all the known differential systems. One of the main applications of the Lie theory of symmetry groups for differential equations is to get group-invariant solutions. Via any subgroup of the symmetry group, the original equation can be reduced to an equation with fewer independent variables by solving the characteristic equation. In general, to each s-parameter subgroup of the full symmetry group, there will correspond a family of group-invariant solutions. Since there are almost always an infinite number of such subgroups, it is usually not feasible to list all possible group-invariant solutions of the system. That needs an effective, systematic means of classifying these solutions, leading to an optimal system of group-invariant solutions from which every other such solution can be derived. About the optimal systems, a lot of excellent work has been done by many famous experts [3 – 7] and some examples of optimal systems can also be found in Ibragimov [8]. Up to now, several methods have been developed to construct optimal systems. The adjoint representation of a Lie group on its Lie algebra was also known to Lie. Its use in classifying group-invariant solutions appeared in [3] and [4] which are written by Ovsiannikov and Olver, respectively. The latter reference contains more details on how to perform the classification of subgroup under the adjoint action.

The Euler equation is one of the basic equations in many physical fields such as fluids, plasmas, condensed matter, astrophysics, and oceanic and atmospheric dynamics. Euler equations are the limit cases of the Navier-Stokes equation for a large Reynolds number, which has been recognized as the basic equation and the very starting point of all problems in fluid physics [9]. Recently, more and more mathematicians and physicists devote lots of efforts to investigate the models of atmospheric and oceanic dynamics. Because Navier-Stokes and Euler equation are starting point of all problems in fluid physics and mechanics, they are the basic models of atmospheric and oceanic dynamics and have been reported by a large number of related papers [10 – 13]. More recently, Lou et al. obtain analytical and exact forms of the vortices and circumfluence of the two-dimensional fluid by means of the general symmetry group theorem [13]. An approximate analytical expression for the (2+1)-dimensional stream function of Katrina 2005 is obtained, in which some messages including the eye size, the hurricane size, the strength, the relation between the hurricane center and the steering flow, etc. are shown. However, Lou et al. only obtained a typhoon solution for barotropic atmosphere, which can only describe the vertical mean atmospheric movement, but can not indicate the 3-dimensional movement and the cold or warm airs distribution in the atmosphere. The simplest baroclinic atmosphere model is the two-layer model, which has been widely used for theoretical research [14]. Here, we will use Olver's method which only depends on fragments of the theory of Lie algebras to construct the optimal system of the (2+1)-dimensional two-layer model equations in atmosphere, in which vertical shear forces are consider:

$$\frac{\partial}{\partial t}\nabla^2\phi + f_0[J(\phi,\nabla^2\phi)+J(h,\nabla^2 h)] = 0, \quad (1)$$

$$\begin{aligned}&\frac{\partial}{\partial t}(\nabla^2 h - f_0c_1h)\\ &+ f_0[J(\phi,\nabla^2 h - f_0c_1h)+J(h,\nabla^2\phi)] = 0,\end{aligned} \quad (2)$$

where operators $J(a,b) = a_xb_y - a_yb_x$ and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, f_0 is a constant and c_1 is a parameter relative to the atmospheric static stability. ϕ denotes the barotropic flow and h means the baroclinic flow, where $\phi = \frac{1}{2}(\phi_1+\phi_2)$ and $h = \frac{1}{2}(\phi_1-\phi_2)$, in which ϕ_1 is the upper level geopotential height, and ϕ_2 is the lower level geopotential height. It is worthy to notice that this two-layer model omits the β-term because our aim is to find vortex solutions (such as TC) and a tropical cyclone (TC) usually appears at low latitudes, where the Coriolis parameter f is small and β-effect is not apparent.

The symmetry group of the two-layer model in atmosphere is studied and its corresponding group invariant solutions are constructed. Ignoring the discussion of the infinite-dimensional subalgebra, we construct an optimal system of one-dimensional and two-dimensional group invariant solutions. Furthermore, using the associated vector fields of the obtained symmetry, we give out the reductions by one-dimensional and two-dimensional subalgebras, and some explicit solutions of the two-layer model equations are obtained. For some interesting solutions, the figures are given out to show their properties. The solutions for tropical cyclones, especially with a double-eyewall structure are constructed and simulated in figures.

This paper is arranged as follows: in Section 2, by using the classical Lie symmetry method, we get the vector fields of the (2+1)-dimensional two-layer model equations (1) and (2). Then the transformations leaving the solutions invariant, i.e. its symmetry groups, are obtained. In Section 3, using the associated vector of the obtained symmetry in Section 2, we construct the one- and two-parameter optimal systems of group-invariant solutions. Based on these optimal systems, some reductions and solutions of (1) and (2) are derived thanks to the Maple symbolic computation. Finally, some conclusions and discussions are given in Section 4.

2. Symmetry Group of Two-Layer Model Equations

By applying the classical Lie symmetry method, we consider the one-parameter group of infinitesimal transformations in (x,y,t,ϕ,h) of (1) and (2) given by

$$\begin{aligned}x^* &= x + \varepsilon\xi(x,y,t,\phi,h) + o(\varepsilon^2),\\ y^* &= y + \varepsilon\eta(x,y,t,\phi,h) + o(\varepsilon^2),\\ t^* &= t + \varepsilon\tau(x,y,t,\phi,h) + o(\varepsilon^2),\\ \phi^* &= \phi + \varepsilon\Phi(x,y,t,\phi,h) + o(\varepsilon^2),\\ h^* &= h + \varepsilon H(x,y,t,\phi,h) + o(\varepsilon^2),\end{aligned} \quad (3)$$

where ε is the group parameter. It is required that (1) and (2) be invariant under the transformations (3), and this yields a system of overdetermined, linear equations for the infinitesimals ξ, η, τ, Φ, and H. Solving these equations, one can find

$$\begin{aligned}&\xi = -C_3yt - C_4y + f_1(t), \quad \eta = C_3xt + C_4x + f_2(t),\\ &\tau = C_1t + C_2,\\ &\Phi = -C_1\phi + \frac{1}{2}C_3f_0(x^2+y^2) + f_0f_2'(t)x\\ &\qquad - f_0f_1'(t)y + f_3(t),\\ &H = -C_1h + C_5,\end{aligned}$$

where $C_i (i = 1,2,\cdots,5)$ are arbitrary constants and $f_1(t)$, $f_2(t)$, and $f_3(t)$ are arbitrary functions of t. The associated vector fields for the one-parameter Lie group of infinitesimal transformations are $v_1, v_2, \cdots, v_8$, given by

$$\begin{aligned}
&v_1 = \partial_t, \quad v_2 = f_3(t)\partial_\phi, \quad v_3 = \partial_h,\\
&v_4 = -y\partial_x + x\partial_y, \quad v_5 = t\partial_t - \phi x\partial_\phi - h\partial_h,\\
&v_6 = -yt\partial_x + xt\partial_y + \frac{1}{2}f_0(x^2+y^2)\partial_\phi,\\
&v_7 = f_1(t)\partial_x - f_0 y f_1'(t)\partial_\phi,\\
&v_8 = f_2(t)\partial_y + f_0 x f_2'(t)\partial_\phi.
\end{aligned} \tag{4}$$

Equation (4) show that the following transformations (defined by $\exp(\varepsilon v_i)$, $i = 1, 2, \cdots, 8$) of variables (x, y, t, ϕ, h) leave the solutions of (1) and (2) invariant:

$$\begin{aligned}
&\exp(\varepsilon v_1) : (x, y, t, \phi, h) \mapsto (x, y, t+\varepsilon, \phi, h),\\
&\exp(\varepsilon v_2) : (x, y, t, \phi, h) \mapsto (x, y, t, \phi, h+\varepsilon),\\
&\exp(\varepsilon v_3) : (x, y, t, \phi, h) \mapsto (x\cos(\varepsilon) - y\sin(\varepsilon),\\
&\qquad x\sin(\varepsilon) + y\cos(\varepsilon), t, \phi, h),\\
&\exp(\varepsilon v_4) : (x, y, t, \phi, h) \mapsto (x, y, \mathrm{e}^{\varepsilon}t, \mathrm{e}^{-\varepsilon}\phi, \mathrm{e}^{-\varepsilon}h),\\
&\exp(\varepsilon v_5) : (x, y, t, \phi, h) \mapsto (x\cos(t\varepsilon) - y\sin(t\varepsilon),\\
&\qquad x\sin(t\varepsilon) + y\cos(t\varepsilon), t,\\
&\qquad \phi + \frac{1}{2}f_0(x^2+y^2)\varepsilon, h),\\
&\exp(\varepsilon v_6) : (x, y, t, \phi, h) \mapsto (x, y, t, \phi + f_3(t)\varepsilon, h),\\
&\exp(\varepsilon v_7) : (x, y, t, \phi, h) \mapsto (x + f_1(t)\varepsilon, y, t,\\
&\qquad \phi - f_0 f_1'(t) y\varepsilon, h),\\
&\exp(\varepsilon v_8) : (x, y, t, \phi, h) \mapsto (x, y + f_2(t)\varepsilon, t,\\
&\qquad \phi + f_0 f_2'(t) x\varepsilon, h).
\end{aligned} \tag{5}$$

Then the following theorem holds:

Theorem 1. If $\phi = p(x, y, t)$, $h = q(x, y, t)$ is a solution of (1) and (2), then so are

$$\begin{aligned}
&\phi^{(1)} = p(x, y, t-\varepsilon), \quad h^{(1)} = q(x, y, t-\varepsilon),\\
&\phi^{(2)} = p(x, y, t), \quad h^{(2)} = q(x, y, t) + \varepsilon,\\
&\phi^{(3)} = p(x\cos(\varepsilon) + y\sin(\varepsilon), -x\sin(\varepsilon) + y\cos(\varepsilon), t),\\
&h^{(3)} = q(x\cos(\varepsilon) + y\sin(\varepsilon), -x\sin(\varepsilon) + y\cos(\varepsilon), t),\\
&\phi^{(4)} = \mathrm{e}^{-\varepsilon} p(x, y, \mathrm{e}^{-\varepsilon}t), \quad h^{(4)} = \mathrm{e}^{-\varepsilon} q(x, y, \mathrm{e}^{-\varepsilon}t),\\
&\phi^{(5)} = p(x\cos(t\varepsilon) + y\sin(t\varepsilon), -x\sin(t\varepsilon) + y\cos(t\varepsilon), t),\\
&h^{(5)} = q(x\cos(t\varepsilon) + y\sin(t\varepsilon), -x\sin(t\varepsilon) + y\cos(t\varepsilon), t)\\
&\qquad + \frac{1}{2}f_0(x^2+y^2)\varepsilon,\\
&\phi^{(6)} = p(x, y, t), \quad h^{(6)} = q(x, y, t) + f_3(t)\varepsilon,\\
&\phi^{(7)} = p(x - f_1(t)\varepsilon, y, t) - f_0 f_1'(t) y\varepsilon,\\
&h^{(7)} = q(x - f_1(t)\varepsilon, y, t),\\
&\phi^{(8)} = p(x, y - f_2(t)\varepsilon, t) + f_0 f_2'(t) x\varepsilon,\\
&h^{(8)} = q(x, y - f_2(t)\varepsilon, t).
\end{aligned}$$

In [15], Clarkson and Kruskal (CK) introduced a direct method to derive symmetry reductions of a nonlinear system without using any group theory. For many types of nonlinear systems, the method can be used to find all the possible similarity reductions. Then Lou and Ma modified CK's direct method [16] to find out the generalized Lie and non-Lie symmetry groups of differential equations by an ansatz reading

$$u(x, y, t) = \alpha(x, y, t) + \beta(x, y, t)U(\xi, \eta, \tau), \tag{6}$$

where ξ, η, τ are all functions of x, y, t. Equation (6) also points out that if $U(x, y, t)$ is a solution of the original differential equation, so is $u(x, y, t)$. Actually, instead of the ansatz (6), the general one-parameter group of symmetries can be obtained by considering the linear combination $a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 + a_6v_6 + a_7v_7 + a_8v_8$ of the given vector fields. But the explicit formulae for the above transformations are very complicated. Factually, it can be represented uniquely in the form

$$\begin{aligned}
g = {}&\exp(\varepsilon_1 v_1) \cdot \exp(\varepsilon_2 v_2) \cdot \exp(\varepsilon_3 v_3) \cdot \exp(\varepsilon_4 v_4)\\
&\cdot \exp(\varepsilon_5 v_5) \cdot \exp(\varepsilon_6 v_6) \cdot \exp(\varepsilon_7 v_7) \cdot \exp(\varepsilon_8 v_8).
\end{aligned}$$

3. Reductions and Solutions of Two-Layer Model Equations

By exploiting the generators v_i of the Lie-point transformations in (4), one can build up exact solutions of (1) and (2) via the symmetry reduction approach. This allows one to lower the number of independent variables of the system of differential equations under consideration using the invariants associated with a given subgroup of the symmetry group. In the following we present some reductions leading to exact solutions of the equations of possible physical interest.

Firstly, we construct an optimal system to classify the group-invariant solutions of (1) and (2). As it is said in [4], the problem of classifying group-invariant solutions reduces to the problem of classifying subgroups of the full symmetry group under conjugation. And the problem of finding an optimal of subgroups is equivalent to that of finding an optimal system of subalgebras. Here, by using the method presented in [3 – 4], we will construct the optimal system of one-dimensional subalgebras of (1) and (2).

From (4), ignoring the discussion of the infinite-dimensional subalgebra v_7 and v_8, and take $f_3(t) = C_6$, one can get the following six operators:

$$v_1 = \partial_t, \quad v_2 = \partial_\phi, \quad v_3 = \partial_h, \quad v_4 = -y\partial_x + x\partial_y,$$
$$v_5 = t\partial_t - \phi x\partial_\phi - h\partial_h,$$
$$v_6 = -yt\partial_x + xt\partial_y + \frac{1}{2}f_0(x^2+y^2)\partial_\phi.$$

Applying the commutator operators $[v_m, v_n] = v_m v_n - v_n v_m$, we get the following table (the entry in row i and the column j representing $[v_i, v_j]$):

Lie	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	0	0	0	v_1	v_4
v_2	0	0	0	0	$-v_2$	0
v_3	0	0	0	0	$-v_3$	0
v_4	0	0	0	0	0	0
v_5	$-v_1$	v_2	v_3	0	0	v_6
v_6	$-v_4$	0	0	0	$-v_6$	0

Therefore, there is

Proposition 1. The operators v_i $(i = 1, 2, \cdots, 6)$ form a Lie algebra, which is a six-dimensional symmetry algebra.

To compute the adjoint representation, we use the Lie series in conjunction with the above commutator table. Applying the formula

$$\mathrm{Ad}\,(\exp(\varepsilon v))\,v_0 = v_0 - \varepsilon[v, v_0] + \frac{1}{2}\varepsilon^2\,[v,[v,v_0]] - \cdots,$$

we can construct the following table:

Ad	v_1	v_2	v_3	v_4	v_5	v_6
v_1	v_1	v_2	v_3	v_4	$v_5 - \varepsilon v_1$	$v_6 - \varepsilon v_4$
v_2	v_1	v_2	v_3	v_4	$v_5 + \varepsilon v_2$	v_6
v_3	v_1	v_2	v_3	v_4	$v_5 + \varepsilon v_3$	v_6
v_4	v_1	v_2	v_3	v_4	v_5	v_6
v_5	$\mathrm{e}^{\varepsilon}v_1$	$\mathrm{e}^{-\varepsilon}v_2$	$\mathrm{e}^{-\varepsilon}v_3$	v_4	v_5	$\mathrm{e}^{-\varepsilon}v_6$
v_6	$v_1 + \varepsilon v_4$	v_2	v_3	v_4	$v_5 + \varepsilon v_6$	v_6

with the (i,j)-th entry indicating $\mathrm{Ad}(\exp(\varepsilon v_i))v_j$.

Following Ovsiannikov [3], one calls two subalgebras v_2 and v_1 of a given Lie algebra equivalent if one can find an element g in the Lie group so that $\mathrm{Ad}\,g(v_1) = v_2$, where $\mathrm{Ad}\,g$ is the adjoint representation of g on v. Given a non-zero vector

$$v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 + a_6v_6,$$

our task is to simplify as many of the coefficients a_i as possible though judicious applications of adjoint maps to v. In this way, omitting the detailed computation, one can get the following theorem by the complicated computation:

Theorem 2. The operators generate an optimal system S_1.

(a) $v_5 + \alpha v_4$, $a_5 \neq 0$;
(b1) $v_6 + v_1 + \alpha v_2 + \beta v_3$, $a_5 = 0$, $a_6 \neq 0$, $a_1 > 0$;
(b2) $v_6 - v_1 + \alpha v_2 + \beta v_3$, $a_5 = 0$, $a_6 \neq 0$, $a_1 < 0$;
(b3) $v_6 + v_3 + \alpha v_2$, $a_1 = a_5 = 0$, $a_6 \neq 0$, $a_3 \neq 0$;
(b4) $v_6 + v_2$, $a_1 = a_3 = a_5 = 0$, $a_6 \neq 0$, $a_2 > 0$;
(b5) $v_6 - v_2$, $a_1 = a_3 = a_5 = 0$, $a_6 \neq 0$, $a_2 < 0$;
(b6) v_6, $a_1 = a_2 = a_3 = a_5 = 0$, $a_6 \neq 0$;
(c1) $v_1 + v_3 + \alpha v_2$, $a_5 = a_6 = 0$, $a_1 \neq 0$, $a_3 \neq 0$;
(c2) $v_1 + v_2$, $a_3 = a_5 = a_6 = 0$, $a_1 \neq 0$, $a_2 > 0$;
(c3) $v_1 - v_2$, $a_3 = a_5 = a_6 = 0$, $a_1 \neq 0$, $a_2 < 0$;
(c4) v_1, $a_2 = a_3 = a_5 = a_6 = 0$, $a_1 \neq 0$;
(d1) $v_4 + v_3 + \alpha v_2$, $a_1 = a_5 = a_6 = 0$, $a_4 \neq 0$, $a_3 \neq 0$;
(d2) $v_4 + v_2$, $a_1 = a_3 = a_5 = a_6 = 0$, $a_4 \neq 0$, $a_2 > 0$;
(d3) $v_4 - v_2$, $a_1 = a_3 = a_5 = a_6 = 0$, $a_4 \neq 0$, $a_2 < 0$;
(d4) v_4, $a_1 = a_2 = a_3 = a_5 = a_6 = 0$, $a_4 \neq 0$;
(e1) $v_2 + v_3$, $a_1 = a_4 = a_5 = a_6 = 0$, $a_2 \neq 0$, $a_3 \neq 0$;
(e2) v_2, $a_1 = a_3 = a_4 = a_5 = a_6 = 0$, $a_2 \neq 0$;
(f) v_3, $a_1 = a_2 = a_4 = a_5 = a_6 = 0$.

Based on S_1, we construct the two-parameter optimal system S_2 by using the method of [3], which are listed in the appendix. Making use of S_1 and S_2, we will discuss the reductions and solutions of (1) and (2).

3.1. Reductions by One-Dimensional Subalgebras

In this section, we will use S_1 to reduce (1) and (2), and then to get the solution of (1) and (2). For case (b3), from

$$-yt\phi_x + xt\phi_y - \frac{1}{2}f_0(x^2+y^2) - \alpha = 0$$

and

$$-yth_x + xth_y - 1 = 0,$$

one can get $\phi = -\frac{f_0x^2+f_0y^2+2\alpha}{2t}\arctan(\frac{x}{y}) + F(\xi,t)$, and $h = -\frac{1}{t}\arctan(\frac{x}{y}) + G(\xi,t)$, where $\xi = x^2+y^2$. Then (1) and (2) is reduced to

$$f_0(-\xi^2F_{\xi\xi\xi} + \xi tF_{\xi\xi t} - 2\xi F_{\xi\xi} + tF_{\xi t} + F_\xi)$$
$$-2\alpha(\xi F_{\xi\xi\xi} + 2F_{\xi\xi}) - 2\xi G_{\xi\xi\xi} - 4G_{\xi\xi} = 0,$$
$$f_0(-\xi^2G_{\xi\xi\xi} + \xi tG_{\xi\xi t} - 2\xi G_{\xi\xi} + tG_{\xi t} + G_\xi)$$
$$-2\alpha(\xi G_{\xi\xi\xi} + 2G_{\xi\xi}) - 2\xi F_{\xi\xi\xi} - 4F_{\xi\xi} = 0.$$

By solving the above equations, one obtains the solution of (1) and (2)

$$\phi = -\frac{f_0x^2 + f_0y^2 + 2\alpha}{2t} \arctan\left(\frac{x}{y}\right) + \frac{r_1(x^2+y^2)}{t} + f_1(t), \tag{7}$$

$$h = -\frac{1}{t}\arctan\left(\frac{x}{y}\right) + \frac{r_2(x^2+y^2)}{t} + f_2(t), \tag{8}$$

where r_i $(i=1,2)$ are arbitrary constants and $f_i(t)$ $(i=1,2)$ are arbitrary functions.

In case (d1), solving

$$-y\phi_x + x\phi_y - \alpha = 0, \quad -yh_x + xh_y - 1 = 0,$$

it follows

$$\phi = -\alpha \arctan\left(\frac{x}{y}\right) + F(\xi,t),$$
$$h = -\arctan\left(\frac{x}{y}\right) + G(\xi,t),$$

where $\xi = x^2+y^2$. Substituting them into (1) and (2), one comes to

$$4f_0(\xi F_{\xi\xi t} + F_{\xi t}) - 8\alpha(\xi F_{\xi\xi\xi} + 2F_{\xi\xi}) - 8\xi G_{\xi\xi\xi} - 16G_{\xi\xi} = 0,$$
$$4f_0(\xi G_{\xi\xi t} + G_{\xi t}) - 8\alpha(\xi G_{\xi\xi\xi} + 2G_{\xi\xi}) - 8\xi F_{\xi\xi\xi} - 16F_{\xi\xi} = 0.$$

Using the solution of the above equations, we get the solution of (1) and (2)

$$\phi = -\alpha \arctan\left(\frac{x}{y}\right) + f_1(t) + f_2(t)\ln(x^2+y^2) + r_1(x^2+y^2), \tag{9}$$

$$h = -\arctan\left(\frac{x}{y}\right) + f_3(t) + f_4(t)\ln(x^2+y^2) + r_2(x^2+y^2), \tag{10}$$

where r_i $(i=1,2)$ are arbitrary constants and $f_i(t)$ $(i=1,2,3,4)$ are arbitrary functions.

For case (d4), by solving

$$-y\phi_x + x\phi_y = 0, \quad -yh_x + xh_y = 0,$$

it leads to

$$\phi = F(\xi,t), \quad h = G(\xi,t),$$

where $\xi = x^2+y^2$. And the reduced equations are

$$\xi F_{\xi\xi t} + F_{\xi t} = 0, \quad \xi G_{\xi\xi t} + G_{\xi t} = 0,$$

whose solution is

$$F(\xi,t) = f_1(t) + f_2(\xi) + f_3(t)\ln(\xi),$$
$$G(\xi,t) = f_4(t) + f_5(\xi) + f_6(t)\ln(\xi),$$

where f_i $(i=1,\cdots,6)$ are arbitrary functions of the corresponding variable. So we obtain the solution of (1) and (2)

$$\phi = f_1(t) + f_2(x^2+y^2) + f_3(t)\ln(x^2+y^2), \tag{11}$$
$$h = f_4(t) + f_5(x^2+y^2) + f_6(t)\ln(x^2+y^2). \tag{12}$$

For the other cases in S_1, you can also use them to reduce (1) and (2) and get the solutions. Here we don't study any more.

3.2. Reductions by Two-Dimensional Subalgebras

In this section, S_2 will be applied to reduce (1) and (2).

Case 1: $(v_5+\alpha_1 v_4, v_6+v_2)$. Then it comes to

$$-\alpha_1 y\phi_x + \alpha_1 x\phi_y + t\phi_t + \phi = 0,$$
$$-\alpha_1 yh_x + \alpha_1 xh_y + th_t + h = 0,$$
$$-yt\phi_x + xt\phi_y - \frac{f_0}{2}(x^2+y^2) - 1 = 0,$$
$$-yth_x + xth_y = 0,$$

and we have $\phi = -\frac{f_0x^2+f_0y^2+2}{2t}(\alpha_1 ln(t) + \arctan(\frac{x}{y})) + \frac{1}{t}F(\xi)$ and $h = \frac{1}{t}G(\xi)$, where $\xi = x^2+y^2$. So (1) and (2) are reduced to

$$2f_0\xi^2 F_{\xi\xi\xi} + 6f_0\xi F_{\xi\xi} + 4\xi F_{\xi\xi\xi} + 8F_{\xi\xi} + \alpha_1 f_0^2 = 0,$$
$$4f_0\xi^2 G_{\xi\xi\xi} + 12f_0\xi G_{\xi\xi} + 8\xi F_{\xi\xi\xi} - f_0^2\xi G_\xi + 16G_{\xi\xi} - 2f_0 G_\xi - f_0^2 G = 0.$$

Solving such equations, the solution of (1) and (2) follows

$$\begin{aligned}\phi = &-\frac{f_0x^2+f_0y^2+2}{2t}\left(\alpha_1\ln(t) + \arctan\left(\frac{x}{y}\right)\right) \\ &- \frac{1}{4t}f_0(\alpha_1-r_1)(x^2+y^2)\ln(f_0(x^2+y^2)+2) \\ &- \frac{1}{2t}(\alpha_1-r_1)\ln(f_0(x^2+y^2)+2) \\ &- \frac{1}{4t}r_1(f_0(x^2+y^2)+2)\ln(x^2+y^2) \\ &+ \frac{r_2}{t}(x^2+y^2) + \frac{r_3}{t},\end{aligned} \tag{13}$$

$$h = \frac{1}{t} r_4 (f_0(x^2+y^2)+2) \cdot \mathrm{HeunC}\left(0,1,0,-\frac{1}{2},-\frac{1}{2},\frac{1}{2} f_0(x^2+y^2)+1\right), \tag{14}$$

where r_i $(i = 1,2,3,4)$ are arbitrary constants and $\mathrm{HeunC}(a_1,a_2,a_3,a_4,a_5,z)$ is the solution of one of Heun's differential equations

$$\frac{\mathrm{d}^2\omega}{\mathrm{d}z^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-d}\right)\frac{\mathrm{d}\omega}{\mathrm{d}z} + \frac{\alpha\beta z - q}{z(z-1)(z-d)}\omega = 0,$$

which is named after Karl L. W. M. Heun [9 – 12].

Case 2: $(v_5 + \alpha_1 v_4, v_6)$. From

$$\begin{aligned}
&-\alpha_1 y\phi_x + \alpha_1 x\phi_y + t\phi_t + \phi = 0,\\
&-\alpha_1 y h_x + \alpha_1 x h_y + t h_t + h = 0,\\
&-yt\phi_x + xt\phi_y - \frac{1}{2} f_0(x^2+y^2) = 0,\\
&-yt h_x + xt h_y = 0,
\end{aligned}$$

it follows $\phi = -\frac{1}{2t} f_0(x^2+y^2)\left(\alpha_1 \ln(t) + \arctan\left(\frac{x}{y}\right)\right) + \frac{1}{t}F(\xi)$, and $h = \frac{1}{t}G(\xi)$, where $\xi = x^2+y^2$. And (1) and (2) are written as

$$2\xi^2 F_{\xi\xi\xi} + 6\xi F_{\xi\xi} + \alpha_1 f_0 = 0,$$

$$4\xi^2 G_{\xi\xi\xi} + 12\xi G_{\xi\xi} - f_0\xi G_\xi - f_0 G = 0,$$

which have the solution

$$\begin{aligned}
F &= -\frac{1}{4}\alpha_1 f_0 \xi \ln(\xi) + r_1\xi + \frac{r_2}{\xi} + r_3,\\
G &= r_4 \mathrm{BesselJ}\left(2, \sqrt{-f_0\xi}\right) + r_5 \mathrm{BesselY}\left(2, \sqrt{-f_0\xi}\right) + \frac{r_6}{\xi},
\end{aligned}$$

where r_i $(i = 1,2,\cdots,6)$ are arbitrary constants. So one obtains the solution of (1) and (2)

$$\phi = -\frac{f_0}{2t}(x^2+y^2)\left(\alpha_1 \ln(t) + \arctan\left(\frac{x}{y}\right)\right) - \frac{\alpha_1 f_0}{4t}(x^2+y^2)\ln(x^2+y^2) + \frac{r_1(x^2+y^2)}{t} + \frac{r_2}{(x^2+y^2)t} + \frac{r_3}{t}, \tag{15}$$

$$h = \frac{r_4}{t}\mathrm{BesselJ}\left(2, \sqrt{-f_0(x^2+y^2)}\right) + \frac{r_5}{t}\mathrm{BesselY}\left(2, \sqrt{-f_0(x^2+y^2)}\right) + \frac{r_6}{(x^2+y^2)t}. \tag{16}$$

Case 3: $(v_6 + v_1 + \alpha_1 v_2 + \beta_1 v_3, v_4 + v_3 + \alpha_2 v_2)$. Using

$$\begin{aligned}
&-yt\phi_x + xt\phi_y + \phi_t - \frac{f_0}{2}(x^2+y^2) - \alpha_1 = 0,\\
&-yt h_x + xt h_y + h_t - \beta_1 = 0,\\
&-y\phi_x + x\phi_y - \alpha_2 = 0,\\
&-y h_x + x h_y - 1 = 0,
\end{aligned}$$

one can have $\phi = -\alpha_2 \arctan\left(\frac{x}{y}\right) - \frac{\alpha_2}{2}t^2 + \frac{f_0x^2+f_0y^2+2\alpha_1}{2}t + F(\xi)$ and $h = -\arctan\left(\frac{x}{y}\right) - \frac{1}{2}t^2 + \beta_1 t + G(\xi)$, where $\xi = x^2+y^2$. So (1) and (2) are reduced to

$$\begin{aligned}
&4\xi(\alpha_2 F_{\xi\xi\xi} + G_{\xi\xi\xi}) + 8(\alpha_2 F_{\xi\xi} + G_{\xi\xi}) - f_0^2 = 0,\\
&\xi(F_{\xi\xi\xi} + \alpha_2 G_{\xi\xi\xi}) + 2(F_{\xi\xi} + \alpha_2 G_{\xi\xi}) = 0.
\end{aligned}$$

Here f_0 is an arbitrary constant, so $\alpha_2 \neq \pm 1$. Solving the above equations, we have calculated the solution of (1) and (2)

$$\phi = -\alpha_2 \arctan\left(\frac{x}{y}\right) - \frac{\alpha_2}{2}t^2 + \frac{f_0x^2+f_0y^2+2\alpha_1}{2}t + r_1\ln(x^2+y^2) + \frac{\alpha_2 f_0^2}{16(\alpha_2^2-1)}(x^2+y^2)^2 + r_2(x^2+y^2) + r_3, \tag{17}$$

$$h = -\arctan\left(\frac{x}{y}\right) - \frac{1}{2}t^2 + \beta_1 t + r_4\ln(x^2+y^2) - \frac{f_0^2}{16(\alpha_2^2-1)}(x^2+y^2)^2 + r_5(x^2+y^2) + r_6, \tag{18}$$

where r_i $(i = 1,2,\cdots,6)$ are arbitrary constants.

Figures 1 – 6 exhibit the barotrophic and baroclinic modes structure and corresponding velocity of the solutions expressed by (7) – (18), respectively. It is found that all solutions can describe a vortex in layered atmosphere, whether in barotrophic mode or baroclinic mode. According to the real structure of a tropical cyclone, especially mature typhoon, we find that solutions in Figure 1, Figure 3, and Figure 4 can be treated

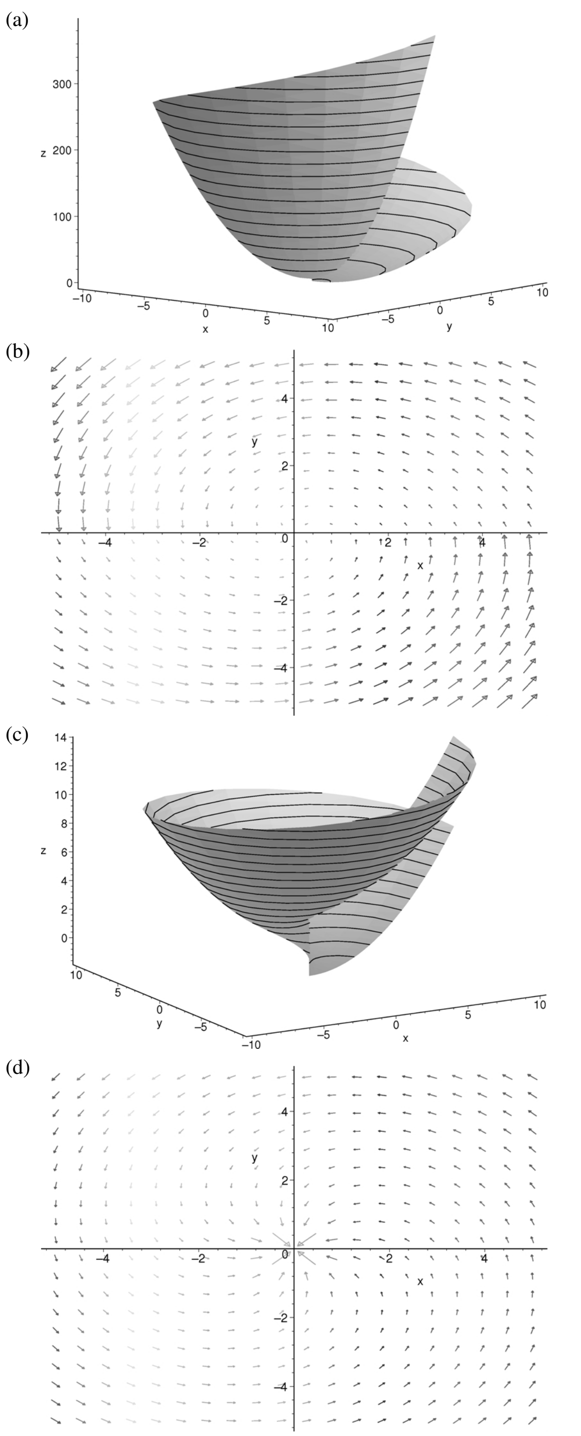

Fig. 1. (a) Structure of function given by (7) with $f_0 = 1$, $\alpha = 1$, $r_1 = 1.5$, $f_1(t) = 0$ at time $t = 1$; (b) plot of velocity related to (a). (c) Structure of function given by (8) with $r_2 = 0.09$, $f_2(t) = 0$ at time $t = 1$; (d) plot of velocity related to (c).

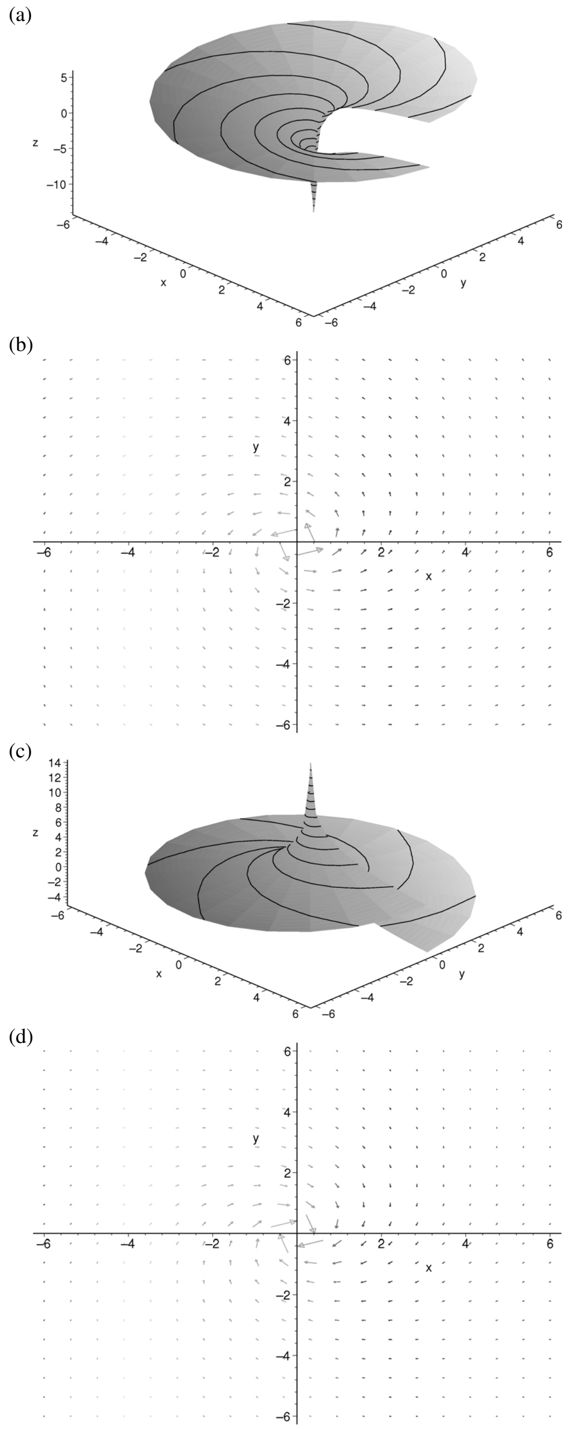

Fig. 2. (a) Structure of function given by (9) with $\alpha = -1$, $r_1 = 0.01$, $f_1(t) = 0$, $f_2(t) = t$ at time $t = 1$; (b) plot of velocity related to (a). (c) Structure of function given by (10) with $r_2 = 0.01$, $f_3(t) = 0$, $f_4(t) = -t$ at time $t = 1$; (d) plot of velocity related to (c).

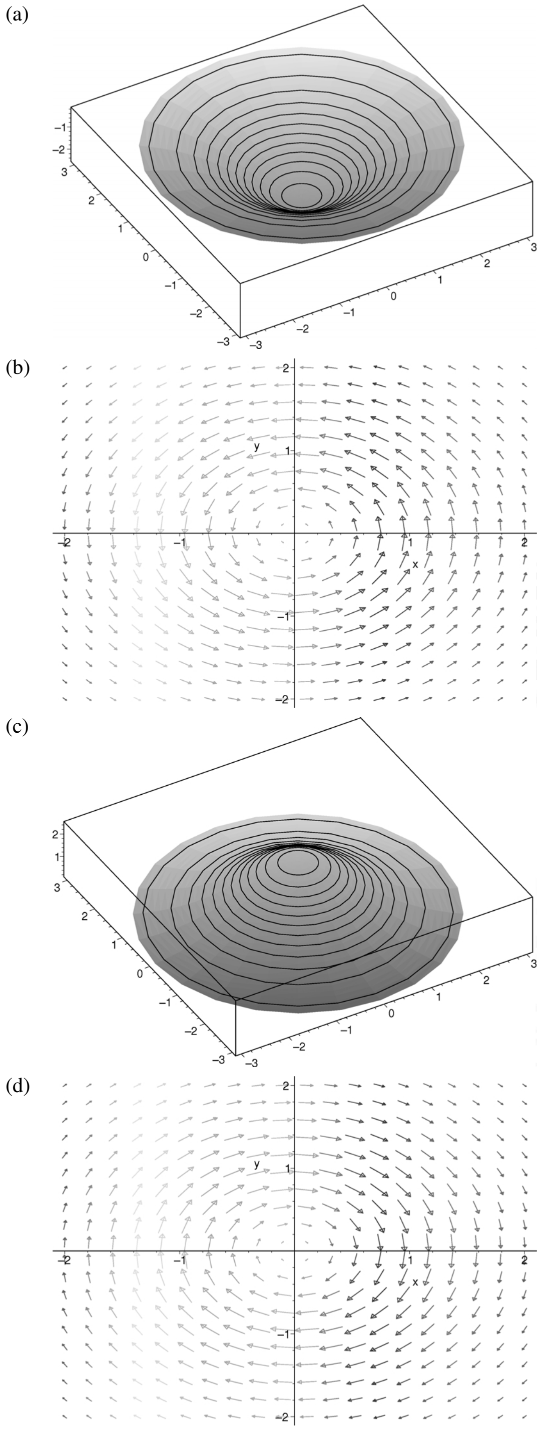

Fig. 3. (a) Structure of function given by (11) with $r_1 = 2$, $r_2 = -10$, $f_1(t) = 0$, $f_3(t) = t - 1$ at time $t = 1$; (b) plot of velocity related to (a). (c) Structure of function given by (12) with $r_3 = 2$, $r_4 = 10$, $f_4(t) = 0$, $f_6(t) = t - 1$ at time $t = 1$; (d) plot of velocity related to (c).

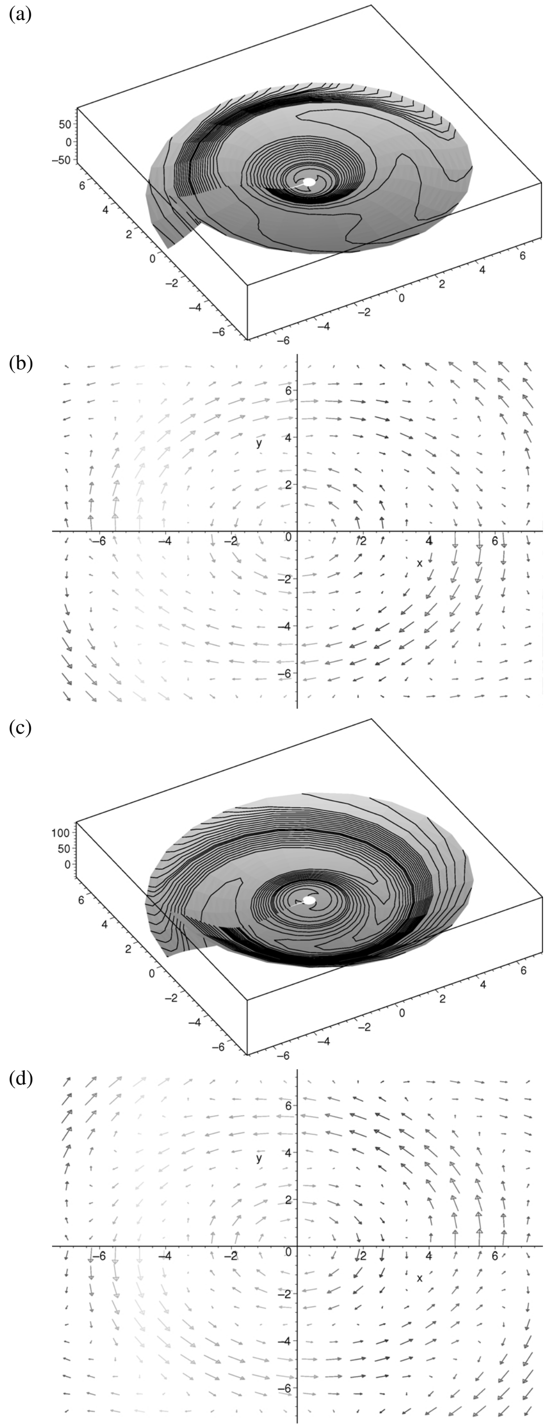

Fig. 4. (a) Structure of function ϕ_1 given by (13) and (14) with $\alpha_1 = 0$, $f_0 = -1$, $r_1 = r_2 = r_3 = 0$, $r_4 = -5$ at time $t = 1$; (b) plot of velocity related to (a). (c) Structure of function ϕ_2 given by (13) and (14) with the same parameters as in (13); (d) plot of velocity related to (c).

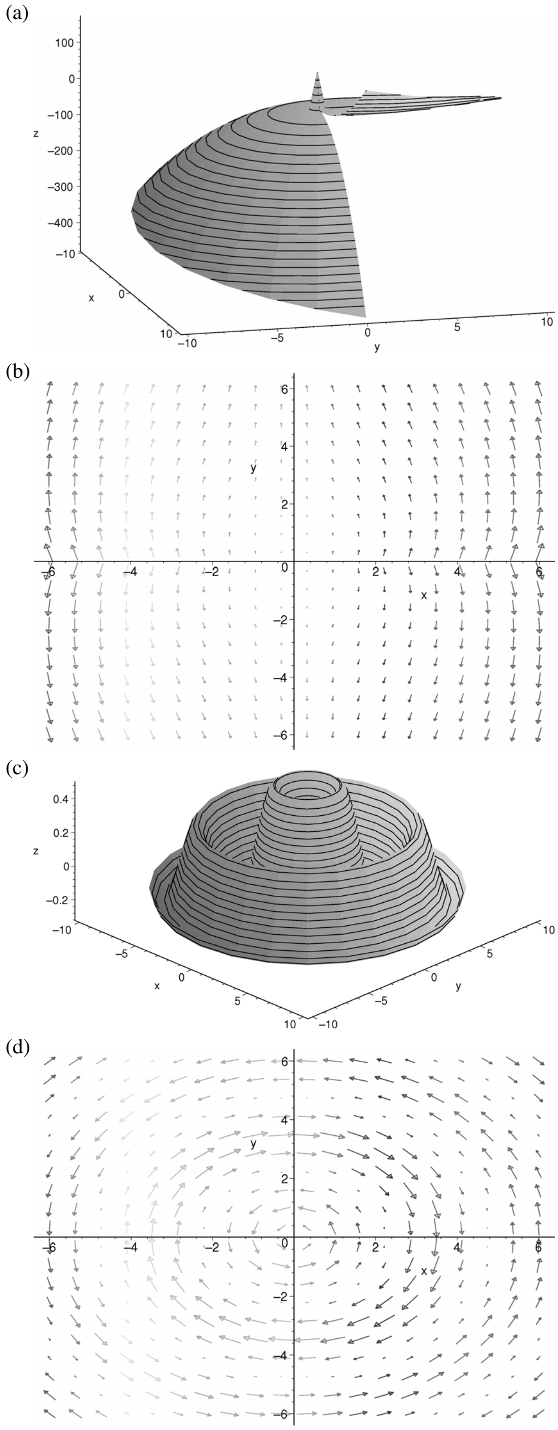

Fig. 5. (a) Structure of function given by (15) with $f_0 = -2$, $r_1 = r_2 = 0.01$, $r_3 = 0$, $\alpha_1 = 0.01$ at time $t = 1$; (b) plot of velocity related to (a). (c) Structure of function given by (16) with $f_0 = -2$, $r_4 = 1$, $r_5 = r_6 = 0$ at time $t = 1$; (d) plot of velocity related to (c).

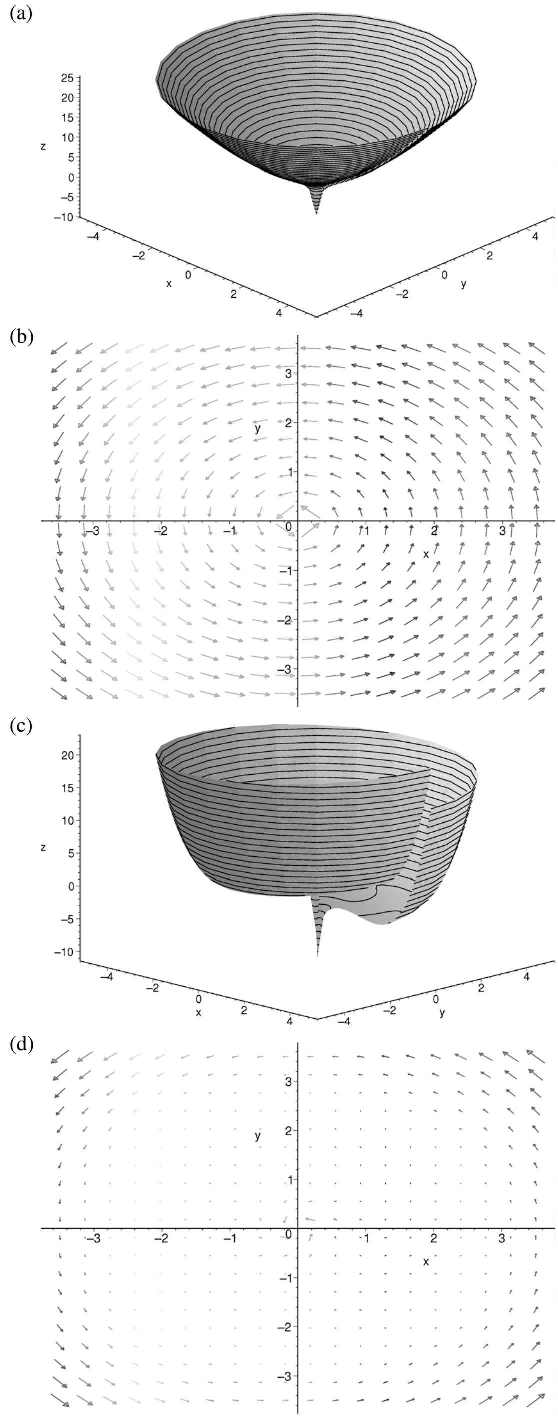

Fig. 6. (a) Structure of function given by (17) with $f_0 = 1$, $\alpha_1 = 0$, $\alpha_2 = 0.1$, $r_1 = 1$, $r_2 = \frac{1}{2}$, $r_3 = 0$ at time $t = 1$; (b) plot of velocity related to (a). (c) Structure of function given by (18) with $f_0 = 1$, $\alpha_2 = 0.1$, $\beta_1 = \frac{1}{2}$, $r_4 = 1$, $r_5 = -1$, $r_6 = 0$ at time $t = 1$; (d) plot of velocity related to (c).

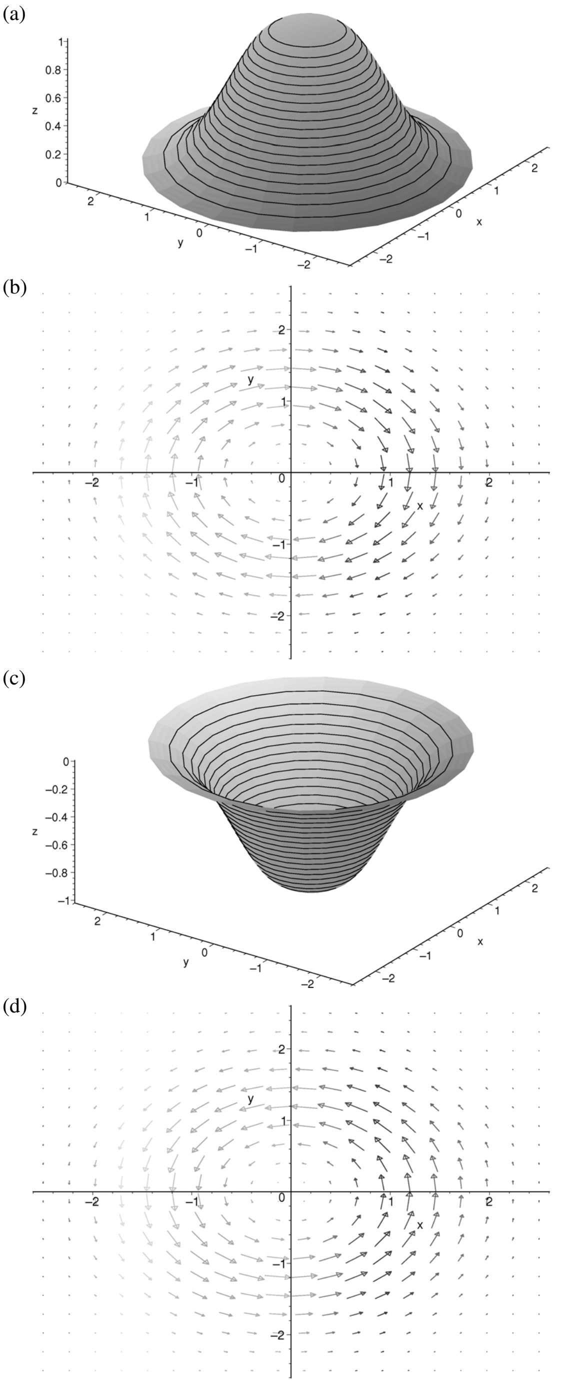

Fig. 7. (a) Structure of function given by (19) with $F_1(t) = 0$, $F_2(x^2+y^2) = (x^2+y^2+1)\mathrm{e}^{-(x^2+y^2)}$ [13], $F_3(t) = t-1$ at time $t = 1$; (b) plot of velocity related to (a). (c) Structure of function given by (20) with $F_4(t) = 0$, $F_5(x^2+y^2) = -(x^2+y^2+1)\mathrm{e}^{-(x^2+y^2)}$, $F_6(t) = t-1$ at time $t = 1$; (d) plot of velocity related to (c).

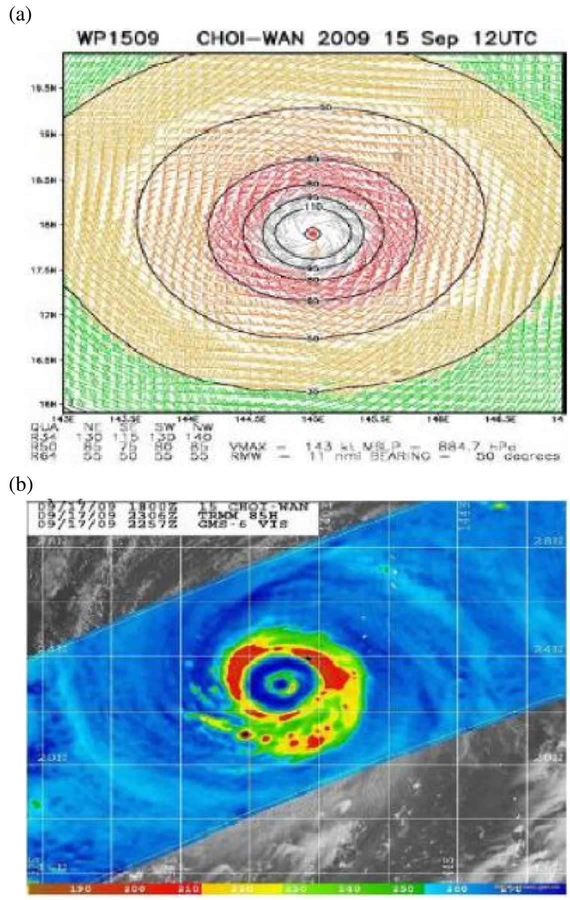

Fig. 8 (colour online). The satellite images of typhoon Choi-Wan on September 15, 2009 (a) and September 17 (b).

as typhoon solutions, possessing typical typhoon structure of eye, eyewall with maximum wind around the eye, and vortex circulation. (7) and (8) give a spiral vortex structure in barotrophic mode (Fig. 1a), similar to the spiral rainband of the typhoon, while its baroclinic structure shows convergence and divergence at lower and upper levels. Equations (11) and (12) express a typical symmetric typhoon structure (Fig. 3) in which the barotrophic structure is similar to that by Lou's typhoon solution [13]. Furthermore, this solution from the two-layer model can give typhoon's baroclinic structure with cyclone at lower level and anticyclone at upper level. To clearly depict the two levels structure, from (11) and (12), we can get the upper level potential ϕ_1 and the lower level potential ϕ_2 as follows:

$$\phi_1 = F_1(t) + F_2(x^2+y^2) + F_3(t)\ln(x^2+y^2), \quad (19)$$

$$\phi_2 = F_4(t) + F_5(x^2+y^2) + F_6(t)\ln(x^2+y^2), \quad (20)$$

where $F_1(t) = f_1(t) + f_4(t)$, $F_2(x^2+y^2) = f_2(x^2+y^2) + f_5(x^2+y^2)$, $F_3(t) = f_3(t) + f_6(t)$, $F_4(t) = f_1(t) - f_4(t)$, $F_5(x^2+y^2) = f_2(x^2+y^2) - f_5(x^2+y^2)$, $F_6(t) = f_3(t) - f_6(t)$. Figure 7 exhibit the structure and velocity of (19)–(20), in which the low-level ϕ_2 and wind field (Fig. 7c,d) can describe the horizontal structure of typhoon Choi-Wan on September 15, 2009 (Fig. 8a).

It is surprising that Figure 4 from (13)–(14) show a double-eyewall structure of the typhoon, although the mechanism of double-eyewall formation is yet an open question. This solution exhibits asymmetry double-eyewall structure in either barotrophic mode or baroclinic mode, very similar to the composed satellite image of the typhoon Choi-Wan on September 17, 2009 (Fig. 8b).

Remark: Double eyewall hurricanes have two concentric rings in which the highest winds are focused. They undergo a cyclical change over portions of the hurricanes life, known as the eyewall replacement cycle. As the outer eyewall develops from a merger of spiral rainbands, it gains strength and saps energy from the inner one. Eventually, the inner eyewall disappears altogether. The outer eyewall then grows smaller and tighter around the eye, and it gains in intensity [17, 18].

4. Conclusions

In summary, we investigate the symmetry of the two-layer model equations by means of the classical Lie symmetry method. The symmetry algebras and group of (1) and (2) are obtained. Specially, the most general one-parameter group of symmetries is given out as the composition of transforms in the eight one-subgroups $\exp(\varepsilon v_1), \exp(\varepsilon v_2), \cdots, \exp(\varepsilon v_8)$ and the most general solution obtainable from a given solution is gained. Next we have classified one- and two-dimensional subalgebras of a Lie algebra of (1) and (2). Then the reductions and some solutions of two-layer model equations by using the associated vector fields of the obtained symmetry are given out. By one-dimensional subalgebras, (1) and (2) is reduced to some (1+1)-dimensional equations and by two-dimensional subalgebras, (1) and (2) is reduced to some ordinary equations. For some interesting explicit solutions of (1) and (2), we also give out figures to show their properties.

It is worth noting that we find the figure which can be used to simulate a double-eyewall structure of a typhoon. As is known, the mechanism of double-eyewall formation is yet an open question. This solution exhibits the asymmetry double-eyewall structure in either barotrophic mode or baroclinic mode, very similar to the composed satellite image of typhoon Choi-Wan on September 17, 2009 (Fig. 8b). Therefore, this solution is worthy to deep investigation later.

Acknowledgements

We would like to thank Prof. Senyue Lou for his enthusiastic guidance and helpful discussions. The work is supported by the National Natural Science Foundation of China (Grant No. 10735030, 40975038 and 90718041), the Program 973 (2006CB403603 and 2005CB422301), Shanghai Leading Academic Discipline Project (No. B412), Program for Changjiang Scholars and Innovative Research Team in University (IRT0734).

Appendix

In this section, we list the two-parameter optimal system S_2: $(v_5+\alpha_1 v_4, v_6+v_3+\alpha_2 v_2)$, $(v_5+\alpha_1 v_4, v_6+v_2)$, $(v_5+\alpha_1 v_4, v_6-v_2)$, $(v_5+\alpha_1 v_4, v_6)$, $(v_5+\alpha_1 v_4, v_1)$, $(v_5+\alpha_1 v_4, v_2)$, $(v_5+\alpha_1 v_4, v_3)$, $(v_5+\alpha_1 v_4, v_2+v_3)$, (v_5, v_4), $(v_6+v_1+\alpha_1 v_2+\beta_1 v_3, v_4+v_3+\alpha_2 v_2)$, $(v_6+v_1+\alpha_1 v_2+\beta_1 v_3, v_4+v_2)$, $(v_6+v_1+\alpha_1 v_2+\beta_1 v_3, v_4-v_2)$, $(v_6+v_1+\alpha_1 v_2+\beta_1 v_3, v_4)$, $(v_6+v_1+\alpha_1 v_2+\beta_1 v_3, v_2+v_3)$, $(v_6+v_1+\alpha_1 v_2+\beta_1 v_3, v_3)$, $(v_6+v_1+\alpha_1 v_2+\beta_1 v_3, v_2)$, $(v_6-v_1+\alpha_1 v_2+\beta_1 v_3, v_4+v_3+\alpha_2 v_2)$, $(v_6-v_1+\alpha_1 v_2+\beta_1 v_3, v_4+v_2)$, $(v_6-v_1+\alpha_1 v_2+\beta_1 v_3, v_4-v_2)$, $(v_6-v_1+\alpha_1 v_2+\beta_1 v_3, v_4)$, $(v_6-v_1+\alpha_1 v_2+\beta_1 v_3, v_2+v_3)$, $(v_6-v_1+\alpha_1 v_2+\beta_1 v_3, v_3)$, $(v_6-v_1+\alpha_1 v_2+\beta_1 v_3, v_2)$, $(v_3+\alpha_1 v_2, v_6+v_2)$, $(v_3+\alpha_1 v_2, v_6-v_2)$, $(v_3+\alpha_1 v_2, v_6)$, $(v_6+v_3+\alpha_1 v_2, v_4+v_3+\alpha_2 v_2)$, $(v_6+v_3+\alpha_1 v_2, v_4+v_2)$, $(v_6+v_3+\alpha_1 v_2, v_4-v_2)$, $(v_6+v_3+\alpha_1 v_2, v_4)$, $(v_6+\alpha_1 v_2, v_2+v_3)$, $(v_6+\alpha_1 v_2, v_3)$, (v_6+v_3, v_2), (v_6, v_2), $(v_6+v_2, v_4+v_3+\alpha_2 v_2)$, (v_6+v_2, v_4+v_2), (v_6+v_2, v_4-v_2), (v_6+v_2, v_4), $(v_6-v_2, v_4+v_3+\alpha_2 v_2)$, (v_6-v_2, v_4+v_2), (v_6-v_2, v_4-v_2), (v_6-v_2, v_4), $(v_6, v_4+v_3+\alpha_2 v_2)$, (v_6, v_4+v_2), (v_6, v_4-v_2), (v_6, v_4), $(v_3+\alpha_1 v_2, v_1+v_2)$, $(v_3+\alpha_1 v_2, v_1-v_2)$, $(v_3+\alpha_1 v_2, v_1)$, $(v_1+v_3+\alpha_1 v_2, v_4+v_3+\alpha_2 v_2)$, $(v_1+v_3+\alpha_1 v_2, v_4+v_2)$, $(v_1+v_3+\alpha_1 v_2, v_4-v_2)$, $(v_1+v_3+\alpha_1 v_2, v_4)$, $(v_1+\alpha_1 v_2, v_2+v_3)$, $(v_1+\alpha_1 v_2, v_3)$, (v_1+v_3, v_2), (v_1, v_2), $(v_1+v_2, v_4+v_3+\alpha_2 v_2)$, (v_1+v_2, v_4+v_2), (v_1+v_2, v_4-v_2), (v_1+v_2, v_4), $(v_1-v_2, v_4+v_3+\alpha_2 v_2)$, (v_1-v_2, v_4+v_2), (v_1-v_2, v_4-v_2), (v_1-v_2, v_4), $(v_1, v_4+v_3+\alpha_2 v_2)$, (v_1, v_4+v_2), (v_1, v_4-v_2), (v_1, v_4), $(v_3+\alpha_1 v_2, v_4+v_2)$, $(v_3+\alpha_1 v_2, v_4-v_2)$, $(v_3+\alpha_1 v_2, v_4)$, $(v_4+\alpha_1 v_2, v_2+v_3)$, $(v_4+\alpha_1 v_2, v_3)$, (v_4+v_3, v_2), (v_4, v_2), (v_3, v_2).

[1] S. Lie, Arch. Math. **6**, 328 (1891).
[2] G. W. Bluman and S. C. Anco, Symmetry and Integration Methods for Differential Equations, Springer, New York 2002.
[3] L. V. Ovsiannikov, Group Analysis of Differential Equations, Academic, New York 1982.
[4] P. J. Olver, Applications of Lie Groups to Differential Equations, Springer-Verlag, New York 1986.
[5] S. V. Coggeshall and J. Meyer-ter-Vehn, J. Math. Phys. **33**, 3585 (1992).
[6] J. Patera, P. Winternitz, and H. Zassenhaus, J. Math. Phys. **16**, 1597 (1975).
[7] K. S. Chou, G. X. Li, and C. Z. Qu, J. Math. Anal. Appl. **261**, 741 (2001).
[8] N. H. Ibragimov, Lie Group Analysis of Differential Equations, CRC Press, Boca Raton 1994.
[9] D. Sundkvist, V. Krasnoselskikh, P. K. Shukla, A. Vaivads, M. André, S. Buchert, and H. Rème, Nature (London) **436**, 825 (2005); G. Pedrizzetti, Phys. Rev. Lett. **94**, 194502 (2005).
[10] H. Lamb, Hydrodynamics, 6th ed., Dover, New York 1945; A. A. Abrashkin and E. I. Yakubovich, Sov. Phys. Dokl. **276**, 370 (1984); G. Gallavotti, Fluid Mechanics. Foundation, see http://ipparco.roma1.infn.it /pagine/ibri.html; G. K. Batchelor, An Introduction to Fluid Dynamics, Cambridge University Press, Cambridge 1967.
[11] R. James Holton, An Introduction to Dynamic Meteorology, Fourth Edition, Elsevier Acad. Press, Amsterdam 2004; Y. Zhang and H. Wu, Numerical Weather Predicction, Beijing Science Press, Beijing 1986.
[12] F. Huang and S. Y. Lou, Phys. Lett. A **320**, 428 (2004); V. L. Saveliev and M. A. Gorokhovski, Phys. Rev. E **72**, 016302 (2005); X. R. Hu, Z. Z. Dong, and Y. Chen, Z. Naturforsch. **65a**, 504 (2010).
[13] S. Y. Lou, M. Jia, X. Y. Tang, and F. Huang, Phys. Rev. E **75** 056318 (2007).
[14] T. G. Shepherd, J. Atmos. Sci. **45** 2014 (1988).
[15] P. A. Clarkson and M. Kruskal, J. Math. Phys. **30**, 2201 (1989).
[16] S. Y. Lou and H. C. Ma, J. Phys. A: Math. Gen. **38**, L129 (2005); H. C. Ma and S. Y. Lou, Z. Naturforsch. **60a**, 229 (2005); H. C. Ma, Chin. Phys. Lett. **22**, 554 (2005); H. C. Ma and S. Y. Lou, Commun. Theor. Phys. (Beijing, China) **46**, 1005 (2006).
[17] K. Blackwell, Coastal Weather Research Center, University of South Alabama http://www.southalabama.edu/cwrc/.
[18] H. E. Willoughby, J. A. Clos, and M. G. Shoreibah, J. Atmos. Sci. **39**, 395 (1982).