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Symmetry analysis and explicit solutions of the (3+1)-dimensional baroclinic potential vorticity equation*

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This paper investigates an important high-dimensional model in the atmospheric and oceanic dynamics-(3+1)-dimensional nonlinear baroclinic potential vorticity equation by the classical Lie group method. Its symmetry algebra, symmetry group and group-invariant solutions are analysed. Otherwise, some exact explicit solutions are obtained from the corresponding (2+1)-dimensional equation, the inviscid barotropic nondivergent vorticy equation. To show the properties and characters of these solutions, some plots as well as their possible physical meanings of the atmospheric circulation are given.

Keywords: (3+1)-dimensional nonlinear baroclinic potential vorticity equation, symmetry group, group-invariant solution, explicit solution

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1. Introduction

Many methods can be used to obtain explicit solutions of nonlinear differential equations, such as inverse scattering transformation, classical and non-classical Lie group approaches, Darboux transformation, Bäcklund transformation, extended Jacobi elliptic function rational expansion method, etc.[1−14] But among these methods, there is no single versatile method which can be used to solve all the differential equations. For lots of differential equations, the classical Lie group or Lie symmetry method[1−4] is a valid and direct way to obtain explicit solutions. Since Sophus Lie[1] set up the theory of Lie group, it has been developed and widely used to find Lie point symmetry algebras and groups for kinds of differential systems. One can use the method to reduce the original equation to another equation with less independent variables. If the original differential equation is ordinary, this technique can also be used to depress its orders. Then theoretically all the corresponding group-invariant solutions can be found by solving the reduced system. However, sometimes one can easily reduce the original equation but hardly obtain explicit solutions of the reduced equation. Especially for the (3+1)-dimensional nonlinear partial differential equation, the analytical solution is very difficult to obtain.

Recently more and more mathematicians and physicists devote lots of effort to investigate the models of atmospheric and oceanic dynamics.[15−17] In the atmospheric and oceanic dynamics, almost all the problems are highly nonlinear. Their corresponding models are always some nonlinear differential equations which are very difficult to solve explicitly. For some meaningful and interesting equations in atmospheric and oceanic dynamics, Lou and his copartners have done a lot of work.[6,18−24] Therein, the (2+1)-dimensional inviscid barotropic nondivergent vorticy (IBNV) equation

$$q = \psi_{xx} + \psi_{yy} \equiv \tilde{\triangle}\psi,$$

$$q_t + [\psi, q] + \beta\psi_x = 0, \quad [\psi, q] = \psi_x q_y - \psi_y q_x$$

has been researched by Huang and Lou,[18] in which some types of general explicit Rossby wave solutions are obtained by using the group theory. Here we investigate the high-dimensional form of the IBNV equa-

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tion called (3+1)-dimensional baroclinic potential vorticity (BPV) equation[25,26] in fluid dynamics,

$$q = \psi_{xx} + \psi_{yy} + \psi_{zz} \equiv \triangle\psi,$$

$$q_t + [\psi, q] + \beta\psi_x = 0, \quad [\psi, q] = \psi_x q_y - \psi_y q_x.$$

In Ref. [26], Zhang *et al.* gave out the approximate solutions of BPV equation. By means of the reductive perturbation method, three types of generalized (2+1)-dimensional Kadomtsev–Petviashvili (KP) equations are also derived from the (3+1)-dimensional BPV equation.[26] In this paper, we will investigate the BPV equation in an alternative way.

First, we analyse its symmetry including the symmetry algebra, symmetry group and the group-invariant solution. Furthermore, in order to obtain more explicit solutions of BPV equation, we utilize some skills on the solutions of (2+1)-dimensional IBNV equation.[18] Thanks to these results, some special explicit interesting solutions are obtained.

The paper is organized as follows. In Section 2, the classical Lie group theory is applied to the (3+1)-dimensional BPV equation. Its Lie point symmetries, the corresponding Lie algebra and Lie group are obtained. One type of group-invariant solution is given out as a case. In Section 3, we obtain some exact explicit solutions of BPV equation based on the results obtained by Huang and Lou. The conclusion will be given in the last section.

2. Symmetry analysis of (3+1)-dimensional BPV equation

The (3+1)-dimensional BPV equation may be written as

$$q = \psi_{xx} + \psi_{yy} + \psi_{zz} \equiv \triangle\psi, \tag{1}$$

$$q_t + [\psi, q] + \beta\psi_x = 0, \quad [\psi, q] = \psi_x q_y - \psi_y q_x, \tag{2}$$

where ψ ($\equiv \psi(x,y,z,t)$) is the dimensionless stream function, q ($\equiv q(x,y,z,t)$) is the vorticity, $\beta = \beta_0(L^2/U)$ and $\beta_0 = (\omega_0/R_0)\cos\phi_0$, in which R_0 is the earth's radius, ω_0 is the angular frequency of the earth's rotation and ϕ_0 is the latitude, L and U are the characteristic horizontal length and velocity scales respectively. The subscripts x, y, z, t represent partial derivatives.

First, we can obtain the Lie point symmetries of the BPV equation by using the classical Lie group method. Applying the classical method to Eqs. (1) and (2), we consider the one-parameter group of infinitesimal transformations in (x, y, z, t, ϕ, q) given by

$$\begin{cases} x^* = x + \epsilon X(x,y,z,t,\psi,q) + o(\epsilon^2), \\ y^* = y + \epsilon Y(x,y,z,t,\psi,q) + o(\epsilon^2), \\ z^* = z + \epsilon Z(x,y,z,t,\psi,q) + o(\epsilon^2), \\ t^* = t + \epsilon T(x,y,z,t,\psi,q) + o(\epsilon^2), \\ \psi^* = \psi + \epsilon \Psi(x,y,z,t,\psi,q) + o(\epsilon^2), \\ q^* = q + \epsilon Q(x,y,z,t,\psi,q) + o(\epsilon^2), \end{cases} \tag{3}$$

where ϵ is group parameter. It is required that equations (1) and (2) are invariant under transformations (3), and this yields a system of overdetermined, linear equations for the infinitesimals X, Y, Z, T, Ψ and Q. Solving these equations, one can have

$$\begin{cases} X = k_1 x + g_3(t), \quad Y = k_1 y + g_4(t), \quad Z = k_1 z + k_2, \quad T = -k_1 t + k_3, \\ \Psi = 3k_1\psi + g_{4t}(t)x - g_{3t}(t)y + g_1(t)z - \frac{1}{2}\beta g_4(t)z^2 + g_2(t) + f(z), \\ Q = k_1 q - \beta g_4(t) + f_{zz}(z), \end{cases} \tag{4}$$

where k_1, k_2, k_3 are arbitrary constants, $f(z)$ is the arbitrary function of z, $g_1(t)$, $g_2(t)$, $g_3(t)$ and $g_4(t)$ are four arbitrary functions of t and subscripts z, t represent derivatives with respect to z, t. Then the corresponding symmetry of ψ and q can be written as

$$\begin{aligned} \sigma^\psi = {} & (k_1 x + g_3(t))\psi_x + (k_1 y + g_4(t))\psi_y + (k_1 z + k_2)\psi_z + (-k_1 t + k_3)\psi_t - (3k_1\psi \\ & + g_{4t}(t)x - g_{3t}(t)y + g_1(t)z - \frac{1}{2}\beta g_4(t)z^2 + g_2(t) + f(z)), \\ \sigma^q = {} & (k_1 x + g_3(t))q_x + (k_1 y + g_4(t))q_y + (k_1 z + k_2)q_z + (-k_1 t + k_3)q_t - (k_1 q - \beta g_4(t) + f_{zz}(z)). \end{aligned}$$

Hence, the corresponding vector field is

$$\begin{aligned} V = {} & (k_1 x + g_3(t))\frac{\partial}{\partial x} + (k_1 y + g_4(t))\frac{\partial}{\partial y} + (k_1 z + k_2)\frac{\partial}{\partial z} + (-k_1 t + k_3)\frac{\partial}{\partial t} + \Big(3k_1\psi + g_{4t}(t)x \\ & - g_{3t}(t)y + g_1(t)z - \frac{1}{2}\beta g_4(t)z^2 + g_2(t) + f(z)\Big)\frac{\partial}{\partial \psi} + (k_1 q - \beta g_4(t) + f_{zz}(z))\frac{\partial}{\partial q}. \end{aligned}$$

Therefore we can say that the symmetry algebra of Eqs. (1) and (2) is generated by the following vector fields

$$\begin{cases} v_1 = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} + z\dfrac{\partial}{\partial z} - t\dfrac{\partial}{\partial t} + q\dfrac{\partial}{\partial q} + 3\psi\dfrac{\partial}{\partial \psi}, \\ v_2 = \dfrac{\partial}{\partial z}, \quad v_3 = \dfrac{\partial}{\partial t}, \\ v_4^f = f(z)\dfrac{\partial}{\partial \psi} + f_{zz}(z)\dfrac{\partial}{\partial q}, \quad v_5^{g_3} = g_3(t)\dfrac{\partial}{\partial x} + g_{3t}(t)y\dfrac{\partial}{\partial \psi}, \\ v_6^{g_4} = g_4(t)\left(\dfrac{\partial}{\partial y} - \beta\dfrac{\partial}{\partial q} - \dfrac{1}{2}\beta z^2\dfrac{\partial}{\partial \psi}\right) + g_{4t}(t)x\dfrac{\partial}{\partial \psi}, \\ v_7^{g_1} = g_1(t)z\dfrac{\partial}{\partial \psi}, \quad v_8^{g_2} = g_2(t)\dfrac{\partial}{\partial \psi}. \end{cases} \tag{5}$$

The commutation relations among these vector fields are given by Table 1, the entry in row i and the column j represents $[v_i, v_j] = v_i v_j - v_j v_i$.

Table 1. The commutation relations $[v_i, v_j]$.

	v_1	v_2	v_3	v_4^f	$v_5^{g_3}$	$v_6^{g_4}$	$v_7^{g_1}$	$v_8^{g_2}$
v_1	0	$-v_2$	v_3	$v_4^{zf_z-3f}$	$v_5^{-g_3-tg_{3t}}$	$v_6^{g_4+tg_{4t}}$	$v_7^{-2g_1-tg_{1t}}$	$v_8^{-3g_2-tg_{2t}}$
v_2	v_2	0	0	$v_4^{f_t}$	0	$v^{-\beta g_{4t}}$	$v_8^{g_1}$	0
v_3	$-v_3$	0	0	0	$v_5^{g_{3t}}$	$v_6^{g_{4t}}$	$v_7^{g_{1t}}$	$v_8^{g_{2t}}$
v_4^f	$v_4^{-zf_z+3f}$	$v_4^{-f_t}$	0	0	0	0	0	0
$v_5^{g_3}$	$v_5^{g_3+tg_{3t}}$	0	$v_5^{-g_{3t}}$	0	0	$v_8^{g_{3t}g_4+g_3g_{4t}}$	0	0
$v_6^{g_4}$	$v_6^{-g_4-tg_{4t}}$	$v^{\beta g_{4t}}$	$v_6^{-g_{4t}}$	0	$v_8^{-g_{3t}g_4-g_3g_{4t}}$	0	0	0
$v_7^{g_1}$	$v_7^{2g_1+tg_{1t}}$	$v_8^{-g_1}$	$v_7^{-g_{1t}}$	0	0	0	0	0
$v_8^{g_2}$	$v_8^{3g_2+tg_{2t}}$	0	$v_8^{-g_{2t}}$	0	0	0	0	0

Since each group G_i is a symmetry group, the above transformations imply that if $\psi = \phi(x,y,z,t), q = p(x,y,z,t)$ are solutions of BPV equation, so there are

$$\begin{aligned}
\tilde{G}_1 : \psi^{(1)} &= \exp(3\epsilon)\phi(x\exp(-\epsilon),\ y\exp(-\epsilon),\ z\exp(-\epsilon), t\exp(\epsilon)), \\
q^{(1)} &= \exp(\epsilon)p(x\exp(-\epsilon), y\exp(-\epsilon), z\exp(-\epsilon), t\exp(\epsilon)); \\
\tilde{G}_2 : \psi^{(2)} &= \phi(x,y,z-\epsilon,t), \quad q^{(2)} = p(x,y,z-\epsilon,t); \\
\tilde{G}_3 : \psi^{(3)} &= \phi(x,y,z,t-\epsilon), \quad q^{(3)} = p(x,y,z,t-\epsilon); \\
\tilde{G}_4 : \psi^{(4)} &= \phi(x,y,z,t) + f(z)\epsilon, \quad q^{(4)} = p(x,y,z,t) + f_{zz}(z)\epsilon; \\
\tilde{G}_5 : \psi^{(5)} &= \phi(x-g_3(t)\epsilon, y, z, t) - g_{3t}(t)y\epsilon, \quad q^{(5)} = p(x-g_3(t)\epsilon, y, z, t); \\
\tilde{G}_6 : \psi^{(6)} &= \phi(x, y-g_4(t)\epsilon, z, t) + \Big(g_{4t}(t)x - \frac{1}{2}\beta z^2 g_4(t)\Big)\epsilon, \\
q^{(6)} &= p(x, y-g_4(t)\epsilon, z, t) - \beta g_4(t)\epsilon; \\
\tilde{G}_7 : \psi^{(7)} &= \phi(x,y,z,t) + g_1(t)z\epsilon, \quad q^{(7)} = p(x,y,z,t); \\
\tilde{G}_8 : \psi^{(8)} &= \phi(x,y,z,t) + g_2(t)\epsilon, \quad q^{(8)} = p(x,y,z,t).
\end{aligned}$$

The general one-parameter group of symmetries is obtained by considering linear combination $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4^f + c_5v_5^{g_3} + c_6v_6^{g_4} + c_7v_7^{g_1} + c_8v_8^{g_2}$ of the given vector fields; the explicit formulae for the transformations are very complicated. Factually, it can be represented uniquely as follows:

$$g = \exp(c_1v_1)\cdot\exp(c_2v_2)\cdot\exp(c_3v_3)\cdot\exp(c_4v_4^f)\cdot\exp(c_5v_5^{g_3})\cdot\exp(c_6v_6^{g_4})\cdot\exp(c_7v_7^{g_1})\cdot\exp(c_8v_8^{g_2}).$$

Thus the most general solutions $\tilde{\psi}$ and $\tilde{q}$ obtained from the given solutions $\psi = \phi(x,y,z,t)$ and $q = p(x,y,z,t)$ are

$$\begin{aligned}\tilde{\psi} &= \phi(X,Y,Z,T) + \frac{c_6 g_{4t}(ct-c_3)x}{c^2} - \frac{c_5 g_{3t}(ct-c_3)y}{c^2} - \frac{c_6\beta g_4(ct-c_3)z^2}{2c^2} + \frac{c_7 g_1(ct-c_3) + c_2 c_6 \beta g_4(ct-c_3)}{c} z \\ &\quad + f\left(\frac{z}{c}-c_2\right)c_4 - c_2 c_7 g_1(ct-c_3) + c_8 g_2(ct-c_3) - \frac{c_5 c_6 g_3(ct-c_3) g_{4t}(ct-c_3)}{c} - \frac{c_2^2 c_6 \beta g_4(ct-c_3)}{2}, \\ \tilde{q} &= p(X,Y,Z,T) + c_4 f_{zz}\left(\frac{z}{c}-c_2\right) - c_6\beta,\end{aligned}$$

and

$$X = \frac{x}{c} - c_5 g_3(ct-c_3), \quad Y = \frac{y}{c} - c_4 g_4(ct-c_3), \quad Z = \frac{z}{c} - c_2, \quad T = ct - c_3,$$

where $c \equiv \exp(c_1)$, $c_1, c_2, \ldots, c_8$ are arbitrary constants, f is the function of $z/c - c_2$ and g_i is the function of $ct - c_3$ $(i = 1, \cdots, 4)$.

The methods used to find group-invariant solutions, generalizing the well-known techniques for finding similarity solutions, provide a systematic computational method for determining large classes of special solutions. These group-invariant solutions are characterized by their invariance under some symmetry group of the system of partial differential equations. Given an s-parameter $(s < p)$ subgroup of the partial differential equations with p-independent variables, one can reduce the original equations to $(p-s)$-dimensional equations.[3] By solving the reduced equations with fewer independent variables, one can obtain group-invariant solutions of the original equations. Similarly, using this method we can reduce the (3+1)-dimensional BPV equation to some lower-dimensional equations, which can be easily realized. For example, we look at a two-dimensional reduction for the BPV equation under v_2 and v_5 in Eq. (5).

For solutions invariant under the translated group generated by $v_2 = \partial/\partial z$, equations (1) and (2) are reduced to their two-dimensional counterparts, which have the same form, but (ψ, q) depend only on (x, y, t). Obviously, the (2+1)-dimensional reduced equations is still too difficult to be solved explicitly, so we look at solutions invariant under a second one-dimensional group v_5. For the $v_5 = \alpha\partial/\partial x + \alpha_t y\, \partial/\partial\psi$ $(\alpha \equiv g_3(t) \neq 0)$, invariants are given by

$$y, \quad t, \quad \tilde{\psi}(y,t) = \psi - \frac{\alpha_t xy}{\alpha}, \quad q = q(y,t)$$

with y, t being independent variables. The reduced system of equations

$$q = \tilde{\psi}_{yy}, \tag{6}$$

$$q_t + \frac{\alpha_t y}{\alpha} q_y + \frac{\alpha_t y \beta}{\alpha} = 0 \tag{7}$$

is readily solved. Then the group-invariant solution of BPV equation can be obtained as

$$\begin{aligned}\psi &= -\frac{\beta y^3}{6} + f_1(t)y + f_2(t) + \frac{xy\alpha_t}{\alpha} \\ &\quad + \iint F_1\left(\frac{y}{\alpha}\right)\mathrm{d}y\mathrm{d}y, \\ q &= -y\beta + F_1\left(\frac{y}{\alpha}\right),\end{aligned}$$

where $f_1(t)$ and $f_2(t)$ are arbitrary functions of t and $F_1(y/\alpha)$ is the arbitrary function of y/α.

In a similar way, one can obtain other three-dimensional or two-dimensional even ordinary differential reduced systems of the BPV equation. However, not all the reduced equations can be solved easily. For the other reduced equations of BPV equation, it is harder to obtain their explicit solutions. Hence it is necessary to looking for different ways to obtain more explicit solutions. In the next section, we will make use of some skills to find some special types of explicit solutions of the (3+1)-dimensional BPV equation from the obtained solutions of its (2+1)-dimensional counterpart.

3. Some explicit solutions from (2+1)-dimensional IBNV equation

When our stream function ψ and vorticity q are independent of z, (3+1)-dimensional BPV equations (1) and (2) becomes the following (2+1)-dimensional equations:

$$q = \psi_{xx} + \psi_{yy} \equiv \tilde{\triangle}\psi, \tag{8}$$

$$q_t + [\psi, q] + \beta\psi_x = 0, \quad [\psi, q] = \psi_x q_y - \psi_y q_x. \tag{9}$$

Equations (8) and (9) can be combined as one equation for substituting Eq. (8) into Eqs. (9) to drop out q,

$$\frac{\partial}{\partial t}\tilde{\triangle}\psi + [\psi, \tilde{\triangle}\psi] + \beta\frac{\partial\psi}{\partial x} = 0, \tag{10}$$

which is called IBNV equation in a beta-plane.[18,27] Huang and Lou[18] have investigated this (2+1)-dimensional IBNV equation by the classical Lie group approach. They gave out its four types of exact explicit similarity solutions by solving the reduced ordinary differential equations. One special type of solution is related to a single periodic Rossby wave. The other three exact solutions including the third-order polynomial solution and an asymptotically periodic Rossby wave solution described by the first types of the Bessel functions show that the stream function may have not only linear shears but also nonlinear shears. Here we utilize two of the four types of solutions to obtain some special explicit solutions of (3+1)-dimensional BPV equation.

3.1. Solutions from third-order polynomial solution

In Ref. [18], an exact third-order polynomial solution of (2+1)-dimensional IBNV equation (10) was obtained,

$$\psi = c_1x^3 + c_2y^3 - \left(3c_2 + \frac{\beta}{2}\right)x^2y - 3c_1xy^2 + C_1(t)x^2 + C_2(t)xy - (C_1(t) + c_0)y^2 + C_3(t)x + C_4(t)y + C_0(t), \quad (11)$$

where c_0, c_1 and c_2 are arbitrary constants and $C_i(t), i = 0, 1, \ldots, 4$ are arbitrary functions of t.

Now we assume that the stream function ψ of (3+1)-dimensional BPV equation has the similar form to Eq. (11) with some modifications, i.e.

$$\begin{aligned}\psi = {}& c_1(z)x^3 + c_2(z)y^3 - \left(3c_2(z) + \frac{\beta}{2}\right)x^2y - 3c_1(z)xy^2 + C_1(t,z)x^2 \\ &+ C_2(t,z)xy - (C_1(t,z) + c_0)y^2 + C_3(t,z)x + C_4(t,z)y + C_0(t,z).\end{aligned} \quad (12)$$

Our aim is to choose the suitable functions of $c_i(z)$ $(i = 0, 1, 2)$ and $C_i(t, z)$ $(i = 0, 1, \ldots 4)$ so that ψ in Eq. (12) is really the solution of (3+1)-dimensional BPV equation.

First, we write down the combination of Eqs. (1) and (2), i.e.

$$\frac{\partial}{\partial t} \triangle \psi + [\psi, \triangle\psi] + \beta\frac{\partial\psi}{\partial x} = 0, \quad (13)$$

where $\triangle\psi = \psi_{xx} + \psi_{yy} + \psi_{zz}$. Then substituting Eq. (12) into Eq. (13), collecting the coefficients of x and y, making them zero, one can obtain some differential equations with respect to $c_i(z)$, $i = 0, 1, 2$ and $C_i(t, z)$, $i = 0, 1, \ldots, 4$:

$$\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_2(z) = 0, \quad \frac{\mathrm{d}^2}{\mathrm{d}z^2}c_1(z) = 0,\\
&-c_1(z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_2(z) + c_2(z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_1(z) = 0,\\
&\frac{\partial^3}{\partial z^2\partial t}C_0(t,z) - C_4(t,z)\frac{\partial^2}{\partial z^2}C_3(t,z) + C_3(t,z)\frac{\partial^2}{\partial z^2}C_4(t,z) = 0,\\
&\frac{\partial^3}{\partial z^2\partial t}C_3(t,z) + C_3(t,z)\frac{\partial^2}{\partial z^2}C_2(t,z) + 2C_1(t,z)\frac{\partial^2}{\partial z^2}C_4(t,z) = 0,\\
&2C_4(t,z)\frac{\partial^2}{\partial z^2}C_1(t,z) + C_2(t,z)\frac{\partial^2}{\partial z^2}C_3(t,z) = 0,\\
&-3C_2(t,z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_1(z) + 6c_2(z)\frac{\partial^2}{\partial z^2}C_1(t,z) - 6C_1(t,z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_2(z) + \beta\frac{\partial^2}{\partial z^2}C_1(t,z) + 3c_1(z)\frac{\partial^2}{\partial z^2}C_2(t,z) = 0,\\
&C_2(t,z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_2(z) + 2c_1(z)\frac{\partial^2}{\partial z^2}C_1(t,z) - 2C_1(t,z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_1(z) + 2c_1(z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_0(z)\\
&\qquad\qquad - 2c_0(z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_1(z) - c_2(z)\frac{\partial^2}{\partial z^2}C_2(t,z) = 0,\\
&2C_1(t,z)\frac{\partial^2}{\partial z^2}C_3(t,z) - C_4(t,z)\frac{\partial^2}{\partial z^2}C_2(t,z) + \frac{\partial^3}{\partial z^2\partial t}C_4(t,z) + C_2(t,z)\frac{\partial^2}{\partial z^2}C_4(t,z)\\
&\qquad\qquad + 2c_0(z)\frac{\partial^2}{\partial z^2}C_3(t,z) - 2C_3(t,z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_0(z) - 2C_3(t,z)\frac{\partial^2}{\partial z^2}C_1(t,z) = 0,\\
&c_1(z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_0(z) - c_0(z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_1(z) - \frac{1}{24}\beta\frac{\partial^2}{\partial z^2}C_2(t,z) = 0,
\end{aligned}$$

$$
\begin{aligned}
&2C_1(t,z)\frac{\partial^2}{\partial z^2}C_2(t,z)-2C_2(t,z)\frac{\partial^2}{\partial z^2}C_1(t,z)+\frac{\partial^3}{\partial z^2\partial t}C_1(t,z)+3c_2(z)\frac{\partial^2}{\partial z^2}C_3(t,z)\\
&\quad+3c_1(z)\frac{\partial^2}{\partial z^2}C_4(t,z)-3C_4(t,z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_1(z)-3C_3(t,z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_2(z)+\frac{1}{2}\beta\frac{\partial^2}{\partial z^2}C_3(t,z)=0,\\
&-6C_1(t,z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_2(z)-12c_0(z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_2(z)+12c_2(z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_0(z)-3C_2(t,z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_1(z)+2\beta\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_0(z)\\
&\quad+2\beta\frac{\partial^2}{\partial z^2}C_1(t,z)+6c_2(z)\frac{\partial^2}{\partial z^2}C_1(t,z)+3c_1(z)\frac{\partial^2}{\partial z^2}C_2(t,z)=0,\\
&-6c_2(z)\frac{\partial^2}{\partial z^2}C_4(t,z)+4c_0(z)\frac{\partial^2}{\partial z^2}C_1(t,z)-6c_1(z)\frac{\partial^2}{\partial z^2}C_3(t,z)+\frac{\partial^3}{\partial z^2\partial t}C_2(t,z)-\beta\frac{\partial^2}{\partial z^2}C_4(t,z)\\
&\quad-6C_3(t,z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_1(z)+6C_4(t,z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_2(z)-4C_1(t,z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_0(z)=0,\\
&-3c_2(z)\frac{\partial^2}{\partial z^2}C_3(t,z)-2C_2(t,z)\frac{\partial^2}{\partial z^2}C_1(t,z)-3c_1(z)\frac{\partial^2}{\partial z^2}C_4(t,z)+2C_1(t,z)\frac{\partial^2}{\partial z^2}C_2(t,z)\\
&\quad+2c_0(z)\frac{\partial^2}{\partial z^2}C_2(t,z)-\frac{\partial^3}{\partial z^2\partial t}C_1(t,z)+3C_3(t,z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_2(z)+3C_4(t,z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_1(z)-2C_2(t,z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_0(z)=0.
\end{aligned}
$$

By solving the above differential equations with the help of *Maple*, one can obtain some complicated results satisfying our requirements. Here we only write down two special statements after disposal.

Result 1

$$
\begin{aligned}
&c_0(z)=k_1+k_2z,\quad c_1(z)=k_3+k_4z,\quad c_2(z)=k_5+k_6z,\\
&C_0(t,z)=s_1(t)+s_2(t)z+p(z),\quad C_1(t,z)=s_3(t)+s_4(t)z,\\
&C_2(t,z)=s_5(t)+s_6(t)z,\quad C_3(t,z)=s_7(t)+s_8(t)z,\quad C_4(t,z)=s_9(t)+s_{10}(t)z.
\end{aligned}
$$

Here k_i $(i=1,\dots,6)$ are arbitrary constants while $s_j(t)$ $(j=1,\dots,10)$ and $p(z)$ are arbitrary functions of t and z, respectively.

Result 2

$$
c_0(z)=c_0(z),\quad c_1(z)=0,\quad c_2(z)=-\frac{\beta}{6},
$$

$$
\begin{aligned}
C_0(t,z)=&\left(z^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_0(z)-4z\frac{\mathrm{d}}{\mathrm{d}z}c_0(z)+6c_0(z)\right)\int t\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}r_1(t)\right)^2\mathrm{d}t\\
&+2\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_0(z)z-2\frac{\mathrm{d}}{\mathrm{d}z}c_0(z)\right)\int t\frac{\mathrm{d}^2}{\mathrm{d}t^2}r_1(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}r_2(t)\mathrm{d}t\\
&+\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_0(z)\int t\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}r_2(t)\right)^2\mathrm{d}t-\frac{4t}{\beta}\int\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_0(z)\left(z\frac{\mathrm{d}}{\mathrm{d}z}F_1(z)+\frac{\mathrm{d}^3}{\mathrm{d}z^3}f_1(z)\right)\mathrm{d}z\\
&+\frac{4\left(\frac{\mathrm{d}}{\mathrm{d}t}r_1(t)-a_1\right)}{\beta}\left[2\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_0(z)\frac{\mathrm{d}}{\mathrm{d}z}c_0(z)-3\int\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_0(z)\right)^2\mathrm{d}z\right]+\frac{2t}{\beta}[zF_1(z)+f_1(z)]\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_0(z)\\
&-\left[\frac{1}{2}t^2B(z,t)^2+\frac{2}{\beta}(zr_4(t)+r_3(t))\right]\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_0(z)+2\left[t^2B(z,t)+\frac{2}{\beta}r_4(t)\right]\frac{\mathrm{d}}{\mathrm{d}z}c_0(z)\\
&+(z^2A_1(t)+zA_2(t))\frac{\mathrm{d}^3}{\mathrm{d}z^3}f_1(z)\\
&-\left[z^2A_1(t)+z(2A_1(t)+A_2(t))\right]\frac{\mathrm{d}^2}{\mathrm{d}z^2}f_1(z)+[3zA_1(t)+2A_1(t)+A_2(t)]\frac{\mathrm{d}}{\mathrm{d}z}f_1(z)\\
&-4A_1(t)f_1(z)+\frac{1}{2}f_2(z)-3t^2\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}r_1(t)\right)^2c_0(z)+\frac{1}{2}zr_5(t)+\frac{1}{2}r_6(t),
\end{aligned}
$$

$$
C_1(t,z)=0,\quad C_2(t,z)=-\frac{1}{t},\quad C_3(t,z)=-\frac{4}{\beta t}\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_0(z)+zt\frac{\mathrm{d}^3}{\mathrm{d}t^3}r_1(t)+t\frac{\mathrm{d}^3}{\mathrm{d}t^3}r_2(t),
$$

$$
C_4(t,z)=2tB(z,t)\frac{\mathrm{d}^2}{\mathrm{d}z^2}c_0(z)-4t\frac{\mathrm{d}^2}{\mathrm{d}t^2}r_1(t)\frac{\mathrm{d}}{\mathrm{d}z}c_0(z)-ztF_1(z)-t\frac{\mathrm{d}}{\mathrm{d}z}f_1(z)+t\frac{\mathrm{d}}{\mathrm{d}t}r_3(t)+zt\frac{\mathrm{d}}{\mathrm{d}t}r_4(t).
$$

Here

$$A_i(t)=\frac{1}{2}t^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}r_i(t)-t\frac{\mathrm{d}}{\mathrm{d}t}r_i(t)+r_i(t)\quad(i=1,2),$$
$$F_1(z)=\frac{\mathrm{d}^3}{\mathrm{d}z^3}f_1(z)-\frac{\mathrm{d}^2}{\mathrm{d}z^2}f_1(z),\quad B(z,t)=z\frac{\mathrm{d}^2}{\mathrm{d}t^2}r_1(t)+\frac{\mathrm{d}^2}{\mathrm{d}t^2}r_2(t)$$

and $r_j(t)$ $(j=1,\dots,6)$, $c_0(z)$, $f_1(z)$, $f_2(z)$ are arbitrary functions of the corresponding independent variables.

Hence the stream function ψ in Eq. (12) with parameters in Result 1 and Result 2 respectively is the explicit solution of Eq. (13). Due to lots of arbitrary functions with respect to z and t in Result 1 and Result 2, the solutions obtained are abundant.

3.2. Solutions from the periodic Rossby wave solution

In Ref. [18], one special type of solution of (2+1)-dimensional IBNV related to a single periodic Rossby wave was also given out. It reads

$$\psi=f(t)y+C\cos\left(kb_0x+ky+kb_0\int f(t)\mathrm{d}t+\frac{\beta b_0t}{k(b_0^2+1)}+x_0\right)+\psi_0(t),\tag{14}$$

where C,k, b_0, x_0 are arbitrary constants and $f(t)$, $\psi_0(t)$ are arbitrary functions of t.

In order to obtain the solutions of Eq. (13) from Eq. (14), one should do some modifications to the parameters in it. Hence we substitute

$$\begin{aligned}\psi=f(t,z)y+C(z)\cos\Bigg(&k(z)b_0(z)x+k(z)y+k(z)b_0(z)\int f(t,z)\mathrm{d}t\\&+\frac{\beta b_0(z)t}{k(z)(b_0(z)^2+1)}+x_0(z)\Bigg)+\psi_0(t,z)\end{aligned}\tag{15}$$

into Eq. (13). Then we will obtain a very complicated equation. Firstly one can collect the coefficients with respect to sin and cos, then collect the coefficients of x and y. To make the obtained coefficients being zero, one can obtain the proper $C(z),k(z),b_0(z),x_0(z)$ and $f(t,z),\psi_0(t,z)$ by solving these differential equations. Because these differential equations are too long, we do not write them down here. We give out three types of the special solutions of Eq. (13) directly.

Solution 1

$$\psi_1=f_1(t)y+(a_2z+a_3)\cos\left(a_4a_5x+a_4y+a_4a_5\int f_1(t)\mathrm{d}t+\frac{\beta a_5t}{a_4(a_5^2+1)}+a_1\right)+\psi_{01}(t),\tag{16}$$

where a_i $(i=1,\dots,5)$ are arbitrary constants, $f_1(t)$ and $\psi_{01}(t)$ are two arbitrary functions of t.

A simple solution describes a typical plane Rossby wave (Fig. 1(a)) with vertical structure of first baroclinic mode, showing out-of-phase stream function pattern between upper and lower levels at z direction (Fig. 1(b)), under parameters of $\psi_{01}(t)=0$, $f_1(t)=-1$, $\beta=1$, $a_1=0$, $a_2=10$, $a_3=0$, $a_4=\pi/2$, $a_5=1$.

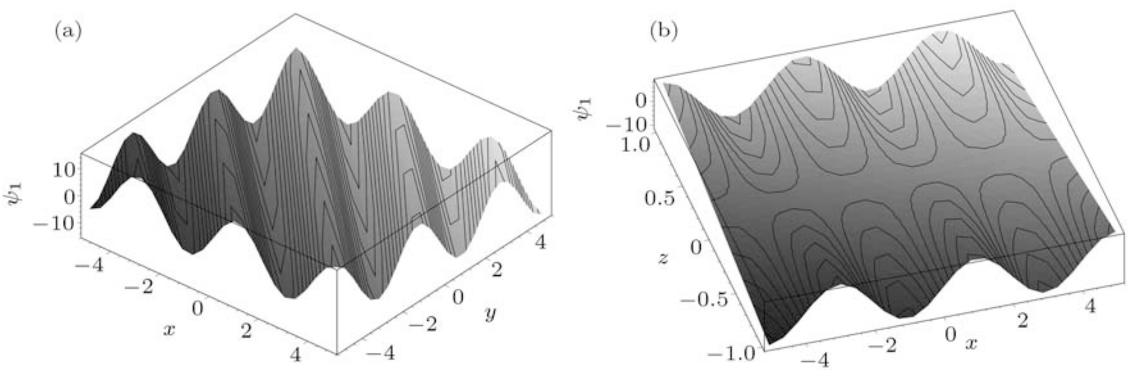


Fig. 1. The ψ_1 given by Eq. (16) with $\psi_{01}(t)=0$, $f_1(t)=-1$, $\beta=1$, $a_1=0$, $a_2=10$, $a_3=0$, $a_4=\pi/2$, $a_5=1$ at (a) $z=1$ and $t=0$; (b) $y=1$ and $t=0$.

Solution 2

$$\psi_2 = f_2(t)y + \sqrt{z^2 + 2b_3 z + b_2 + b_3^2}\exp(b_4)\cos\left(b_5 b_6 x + b_5 y + b_5 b_6 \int f_2(t)\mathrm{d}t + \frac{\beta b_6 t}{b_5(b_6^2+1)} + \arctan\frac{z+b_3}{\sqrt{b_2}} + b_1\right) + \psi_{02}(t), \tag{17}$$

where b_i $(i = 1, \dots, 6)$ are arbitrary constants, $f_2(t)$ and $\psi_{02}(t)$ are two arbitrary functions of t.

Solution 2 in Eq. (17) is an extension of solution 1 in Eq. (16) which still depicts plane Rossby waves, but possesses richer vertical structure patterns. For example, figure 2 is a plot of a special selection for ψ_2 described by Eq. (17) with

$$f_2(t) = -0.5,\quad \psi_{02}(t) = 0,\quad b_1 = 0,\quad b_2 = 1,\quad b_3 = 2,\quad b_4 = 0.5,\quad b_5 = \pi/2,\quad b_6 = 1,\quad \beta = 1. \tag{18}$$

Solution 2 has similar plane Rossby waves pattern (omitted) as that in Fig. 1(a) of solution 1, while the vertical structure could tilt westward (Fig. 2(a)) or southward (Fig. 2(b)) with respect to the increasing height. Parameters b_5 and b_6 control the vertical structure of Rossby wave tilting westward (eastward) or southward (northward) in Eq. (17). It is noticed that the upper and lower stream function (Fig. 2) appears to be asymmetry circulation pattern, not always symmetric as that in Fig. 1(b), which is controlled by parameter b_3.

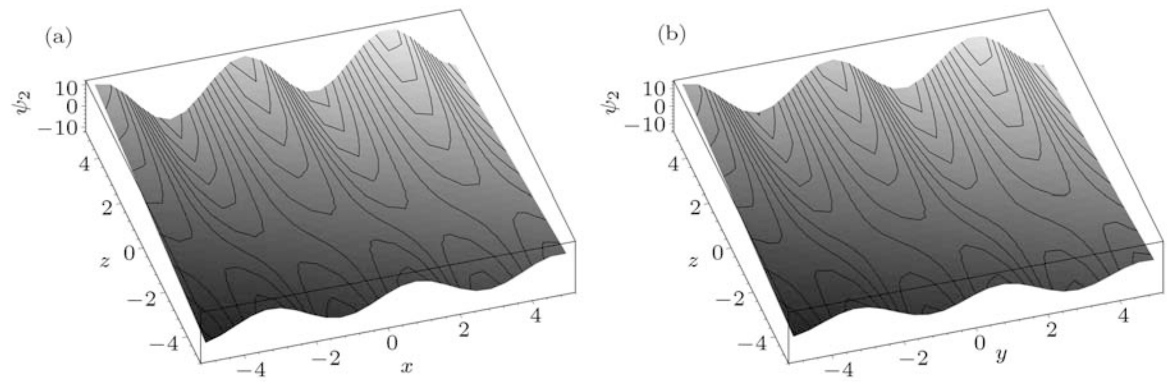
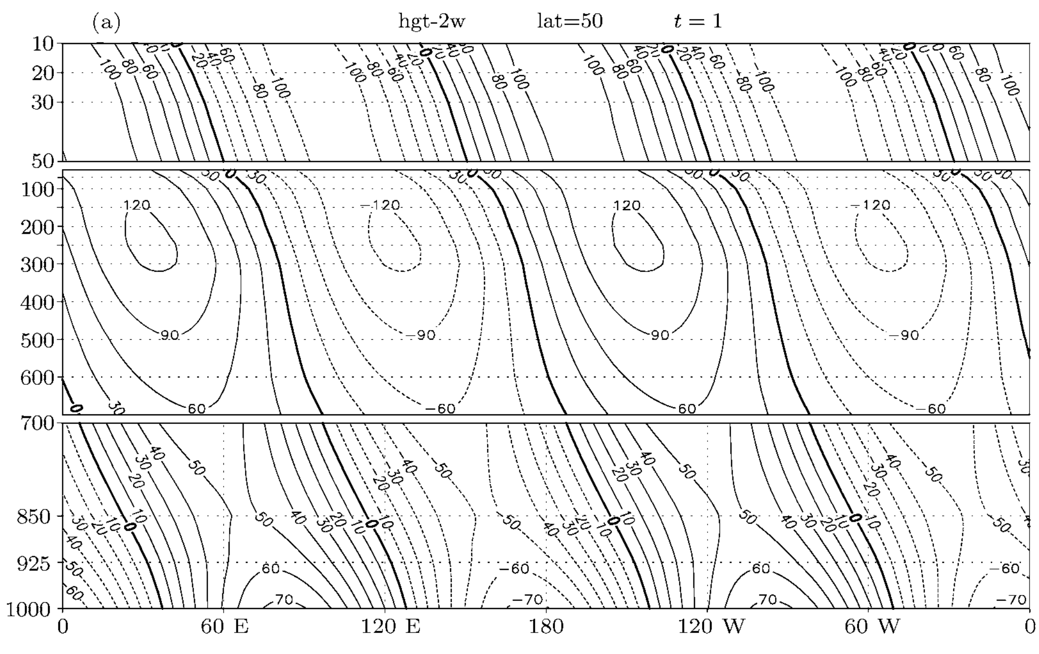


Fig. 2. The ψ_2 given by Eq. (17) with Eq. (18) at (a) $y = 0$ and $t = 0$; (b) $x = 0$ and $t = 0$.



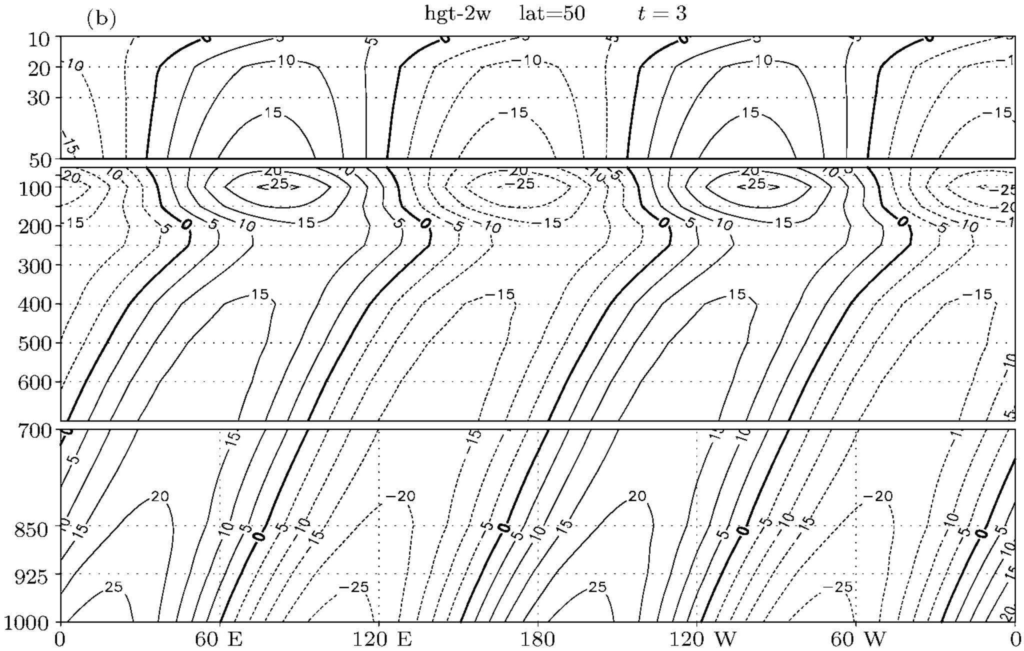


Fig. 3. Longitude-pressure cross section in winter (a) and summer (b) stationary wave distribution expressed by the geopotential height anomaly (in dgpm) in harmonic wavenumber-2 around 50° N.

As we know, the stationary waves always appear to be wavenumber-2 structure in the Northern Hemisphere because of the two-lands-two-oceans (the American and Euro–Asian continents, the North Pacific and the North Atlantic) land sea distribution. Usually geopotential height at mid-high latitudes could be equivalent to the geostrophic stream function because of the geostrophic balance relation. It is found that the climatological wavenumber-2 planetary Rossby waves at 50° N show westward tilting vertical structure in winter (Fig. 3(a)), while eastward tilt in summer (Fig. 3(b)). The intensity of wave amplitude also exhibits asymmetry in the lower troposphere (1000–700 hPa) and upper troposphere (700–100 hPa). The real case is very similar to our theoretical solution in Fig. 2(a).

Solution 3

$$\psi_3 = f_3(t)y + C(z)\cos(k(z)y + x_0(z)) + \psi_{03}(t), \tag{19}$$

where $f_3(t), \psi_{03}(t)$ and $x_0(z), k(z), C(z)$ are arbitrary functions of t and z, respectively. Since ψ_3 is independent of x, this solution only reflects the meridional cells on y–z cross sections at arbitrary time. The traditional zonal mean meridional cells of atmosphere are the well-known three meridional cells: Hadley cell, Ferrel cell and polar cell in the Northern or Southern Hemisphere. Solution 3 certainly can describe this traditional three meridional cells as those in Fig. 4 with

$$f_3(t) = 0\,, \quad \psi_{03}(t) = 0\,, \quad x_0(z) = \pi/2\,,$$
$$k(z) = \pi/1.1\,, \quad C(z) = \sin(\pi z/4 + 0.67)\,.$$

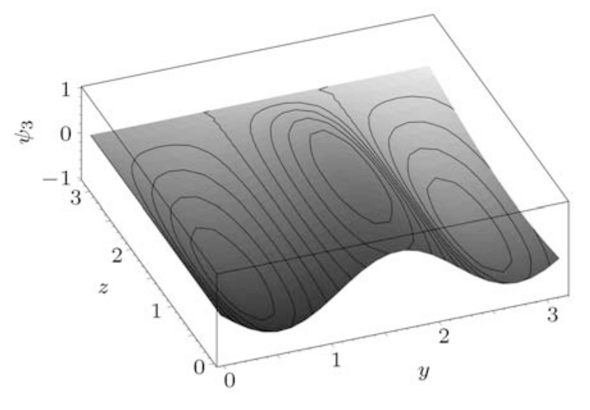


Fig. 4. The ψ_3 given by Eq. (19) with $f_3(t) = 0$, $\psi_{03}(t) = 0$, $x_0(z) = \pi/2$, $k(z) = \pi/1.1$, $C(z) = \sin(\pi z/4 + 0.67)$ at $t = 0$.

Actually, ψ_3 in Eq. (19) has very rich meridional circulation solutions since there are many arbitrary functions. Figure 5 is a plot of four special different selections of the five arbitrary functions in Eq. (19) for ψ_3. There are

Fig. 5(a): $f_3(t) = \cos(t) + 2, \psi_{03}(t) = \sin(t), C(z) = 10, k(z) = \tanh(z), x_0(z) = -z$;

Fig. 5(b): $f_3(t) = \cos(t) + 2, \psi_{03}(t) = 0, C(z) = \exp(z), k(z) = 1, x_0(z) = 0$;

Fig. 5(c): $f_3(t) = \cos(t) + 2, \psi_{03}(t) = 0, C(z) = 10z, k(z) = 1, x_0(z) = z$;

Fig. 5(d): $f_3(t) = \cos(t) + 2, \psi_{03}(t) = 0, C(z) = \tan(z), k(z) = z, x_0(z) = 0$.

Whether the above four special solutions have physical meanings in atmosphere needs further study.

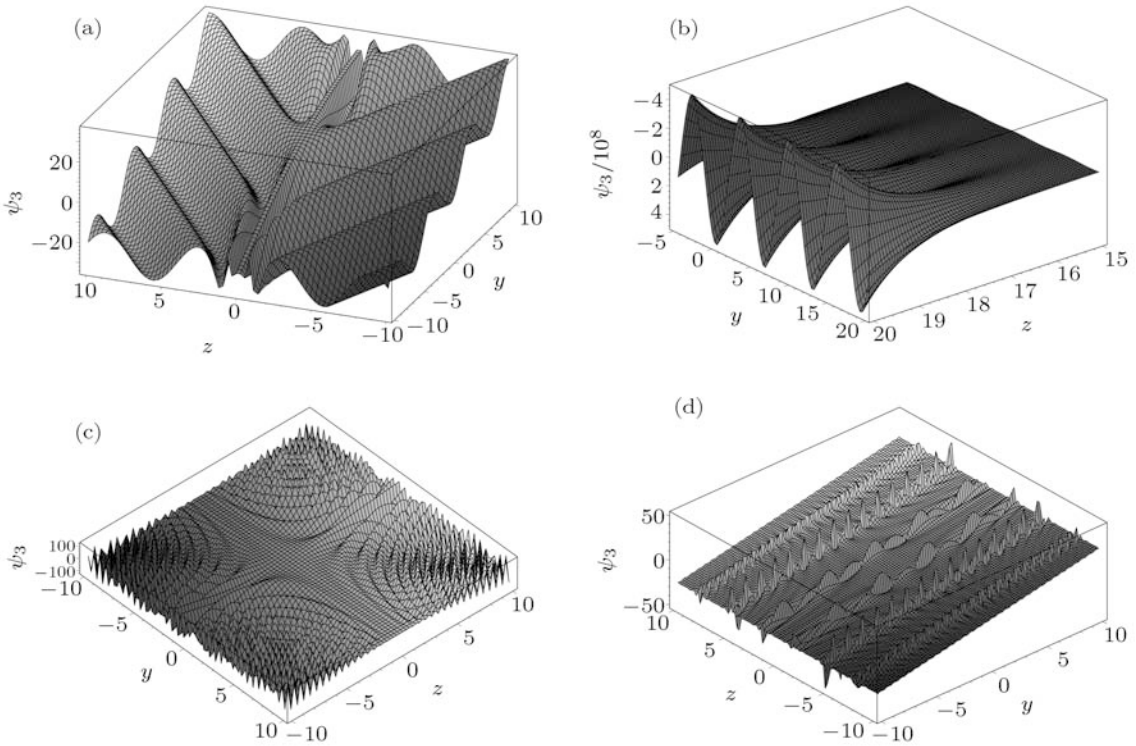


Fig. 5. (a)–(d) ψ_3 given by Eq. (19) with four different selections all at $t = 1$.

4. Conclusion

We know that it is useful and meaningful to investigate the models of atmospheric and oceanic dynamics. But due to their nonlinearity, it is hard to give out their explicit solutions. Especially for the (3+1)-dimensional nonlinear partial differential equation, to gain the explicit solutions is more difficult to realize. In this paper, the (3+1)-dimensional BPV equation in fluid dynamics is studied. First, making use of the classical Lie group method, we analyse the symmetry of the BPV equation including its point Lie symmetries, Lie algebra and the corresponding Lie group. A two-dimensional symmetry reduction is given out and the corresponding group-invariant solution is obtained. Because of the high dimensions of the BPV equation, more group-invariant solutions cannot be obtained easily. Hence looking for another way to obtain more explicit solutions is necessary. The (2+1)-dimensional counterpart of (3+1)-dimensional BPV equation has been studied by Huang and Lou.[18] By the Lie group theory, four types of explicit solutions were gained by them. Thanks to their results, some special explicit solutions of (3+1)-dimensional BPV equation are luckily obtained by adding the independent variable z to the arbitrary parameters in the solutions of (2+1)-dimensional IBNV equation. To show the properties and characters of these solutions, some plots as well as their possible physical meanings of the atmospheric circulation are given out lastly.

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