Commun. Theor. Phys. (Beijing, China) **54** (2010) pp. 615–618
Vol. 54, No. 4, October 15, 2010

An Elastic Interaction Model of Vortices*

JIA Man (贾曼),[1] HUANG Fei (黄菲),[2] LOU Sen-Yue (楼森岳),[1,3,4] SUN Ji-Lin (孙即霖),[2] and TANG Xiao-Yan (唐晓艳)[3]

[1]Faculty of Science, Ningbo University, Ningbo 315211, China

[2]Physical Oceanography Laboratory and Ocean-Atmosphere Interaction and Climate Laboratory, Ocean University of China, Qingdao 266100, China

[3]Department of Physics, Shanghai Jiao Tong University, Shanghai 200240, China

[4]School of Mathematics, Fudan University, Shanghai 200433, China

(Received December 9, 2009)

Abstract *Soliton theory plays an important role in nonlinear physics. The elastic interaction among solitons is one of the most important properties for integrable systems. In this Letter, an elastic vortex interaction model is proposed. It is found that the momenta, vortex momenta and the energies of every one vortex and the interaction energies of every two vortices are conserved.*

PACS numbers: 02.30.Ik, 47.32.-y, 47.53.+n, 47.35.-i, 47.10.ab
Key words: vortices, elastic interactions, Euler equation

It is a well-known fact that the soliton interactions for most of the integrable systems such as the Korteweg-de Vries (KdV),[1] modified KdV (mKdV), nonlinear Schrödinger (NLS), and sine-Gordon (sG) equations are completely elastic.

In $(2+1)$ and $(3+1)$-dimensional cases, as a special type of solitons, vortices play a very important role in almost all the natural scientific fields, such as the bio-x sciences, Earth science, physics, chemistry, optics, fluid physics, and so on.[2−9]

Especially, for a real atmospheric or oceanic system, there are many types of complicated abnormal vortices, which have made increasing destruction for the world over the past 30 years.[10] The hurricanes,[10−11] tornados,[12] atmospheric blockings,[13] subtropical high,[14] polar vortices,[15−16] etc. are all typical vortices. Moreover, several vortices of the same type (e.g. two hurricanes) or different types (say, hurricanes and subtropical high), may simultaneously exist. Thus comes a natural important question:

How to describe the interactions among the vortices?

In this letter, we focus only on the possible elastic interactions among vortices of (2+1)-dimensional rotating fluid systems such as the atmospheric and oceanic systems, which can be described by the nonlinear inviscid nondissipative and equivalent barotropic vorticity equation (NINEBVE) in a beta-plane channel[17]

$$\omega = \psi_{xx} + \psi_{yy}\,, \tag{1}$$

$$\omega_t + [\psi, \omega] + \beta\psi_x = 0\,, \tag{2}$$

where the velocity $\vec{u} = \{u_1, u_2\}$ is determined by the stream function ψ through $u_1 = -\psi_y$, $u_2 = \psi_x$ and the Jacobian operator (or, namely, the commutator) $[A, B]$ is defined as $[A, B] \equiv A_x B_y - B_x A_y$. When $\beta = 0$, the NINEBVE reduces to the known (2+1)-dimensional Euler equation (EE), which has been studied by various authors and many kinds of exact solutions[18] have been found.

In principle, to study the interactions among multiple vortices, one has to solve out an exact multi-vortex solution of Eqs. (1)–(2). However, it is very difficult to find exact analytical multiple vortex solutions for the NINEBVE. In fact, even for the ideal EE case, to find exact analytical multiple vortex solutions is also not easy. Here we try to study the multiple vortices in an alternative way by supposing the following conditions:

(i) Every vortex is localized for the vorticity ω_i while the velocities ψ_{ix}, ψ_{iy} may not be localized.

(ii) Every vortex is an approximate solution of the NINEBVE when the others are far away from it.

(iii) The whole system, i.e., the system with the total vorticity and stream function

$$\omega = \sum_{i=1}^{N} \omega_i\,, \quad \psi = \sum_{i=1}^{N} \psi_i\,, \tag{3}$$

is an exact solution of the NINEBVE.

According to the above assumptions, substituting Eq. (3) into NINEBVE (1)–(2), one can easily obtain the following multiple vortex interaction model (MVIM)

$$\omega_i = \psi_{ixx} + \psi_{iyy}\,, \quad i = 1, 2, \ldots, N\,, \tag{4}$$

$$\omega_{it} + [\psi_i, \omega_i] + \beta\psi_{ix} + \sum_{j=1, j\neq i} [\psi_i, \omega_j] = 0\,. \tag{5}$$

It is clear that when we consider the problem near the i-th vortex $\{\omega_i, \psi_i\}$, the last term of Eq. (5) is small if the other vortices are far away from the i-th vortex because of the localization property of ω_j. In addition, the summation of Eqs. (4) and (5) for i from 1 to N recovers the total system. This fact implies that the system (4)–(5) is

*Supported by the National Natural Science Foundation of China under Grant No. 10735030, the National Basic Research Programs of China (973 Programs) under Grant Nos. 2007CB814800 and 2005CB422301, and the PCSIRT (IRT0734)

equivalent to the following one

$$\omega_i = \psi_{ixx} + \psi_{iyy}, \quad i = 1, 2, \ldots, N-1, \tag{6}$$

$$\omega_{it} + \beta\psi_{ix} + [\psi_i, \omega] = 0, \tag{7}$$

while the vorticity ω of the total system is determined by the decoupled system (1)–(2). From the equivalent system (6)–(7) and (1)–(2), one can conclude that the model is weak integrable because the existence of the weak Lax pair of (1)–(2)[11,19] and the linearity of the remained equations (6)–(7) with respect to the other variables ψ_i and ω_i whence ω is obtained from the decoupled system.

Applying the standard Lie point symmetry approach to the MVIM, it is not difficult to find the model is invariant under the transformation with the vector field

$$\begin{aligned} V &= (cx + x_0)\partial_x + (cy + y_0)\partial_y - (ct + t_0)\partial_t \\ &+ \sum_{i=1}^{N}\Big(3c\psi_i + \psi_{i0}(t) + \sum_{k=1}^{N} c_{ik}\psi_k\Big)\partial_{\psi_i} \\ &+ \sum_{i=1}^{N}\Big(3c\omega_i + \sum_{k=1}^{N} c_{ik}\omega_k\Big)\partial_{\omega_i}, \quad \sum_{i=1}^{N} c_{ik} = 0, \end{aligned} \tag{8}$$

where x_0, y_0, t_0, c, and c_{ik}, i, $k = 1, 2, \ldots, N$ are constants and $\psi_{i0}(t)$ are arbitrary functions of t. Equation (8) shows us that the MVIM is space-time translation invariant (x_0, y_0, and t_0 parts), scaling invariant (c part), time dependent vortex translation ($\psi_{i0}(t)$ parts) and singular ($\det c_{ij} = 0$) rotations of the fields ψ_i and ω_i (c_{ij} parts).

Now, we are interested in that for the MVIM (4)–(5), what kinds of physical quantities will be conserved? Especially, what kind of quantity will be exchanged among different interacting vortices?

A conservation law (CL) means the existence of the conserved density ρ and conserved flows J_1 and J_2 such that

$$\rho_t + J_{1x} + J_{2y} = 0. \tag{9}$$

Obviously, all the quantities, which can be expressed by the differentiations of x and/or y are conserved due to $(A_x)_t = (A_t)_x$. Therefore, the momenta (with the densities $-\psi_{iy}$ and ψ_{ix}) and vorticity momenta (with the density $\omega_i = \psi_{ixx} + \psi_{iyy}$) for every vortex are conserved, because they are really total differentiations of some quantities with respect to x or y.

The important nontrivial conserved quantities are the energies for every vortex with the conserved densities

$$E_i \equiv \frac{1}{2}(u_{i1}^2 + u_{i2}^2) \equiv \frac{1}{2}(\psi_{iy}^2 + \psi_{ix}^2), \tag{10}$$

and fluxes

$$J_{1i} = -\psi_i\psi_{ixt} - \frac{1}{2}\psi_i^2(\beta + \omega_y), \quad J_{2i} = \frac{1}{2}\psi_i^2\omega_x - \psi_i\psi_{iyt}.$$

It is interesting that in addition to the above energies for every vortex, the interaction energies between every two vortices with the conserved densities

$$E_{ij} \equiv \psi_{ix}\psi_{jx} + \psi_{iy}\psi_{jy}, \tag{11}$$

and fluxes,

$$J_{1ij} = -\psi_i\psi_{jxt} - \psi_j\psi_{ixt} - \psi_i\psi_j(\beta + \omega_y),$$

$$J_{2ij} = -\psi_i\psi_{jyt} - \psi_j\psi_{iyt} + \psi_i\psi_j\omega_x,$$

are also conserved. Naturally, the conserved densities E_i and E_{ij} imply the conservation of the total energy with the density,

$$\begin{aligned} E_{\rm total} &\equiv \sum_{i=1}^{N} E_i + \sum_{j<i} E_{ij} \\ &= \frac{1}{2}\Big(\sum_{i=1}^{N}\psi_{ix}\Big)^2 + \frac{1}{2}\Big(\sum_{i=1}^{N}\psi_{iy}\Big)^2. \end{aligned} \tag{12}$$

For the simple EE, it is known that $(1/2)\omega^2$ is the density of the conserved enstrophy. Actually the arbitrary function of the vorticity $f(\omega)$ is a known conserved density of the EE. Formally, for the total system of the NINEBVE, we can also prove that $f \equiv f(\omega + \beta y)$, the arbitrary function of $\omega + \beta y$, is a conserved density, and the related conservation law reads

$$f_t - (f\psi_y)_x + (f\psi_x)_y = 0. \tag{13}$$

However, for the single vortex with $\{\psi_i, \omega_i\}$, the enstrophy, $(\omega_i + \beta y)^2$, is not a conserved quantity. Besides, $f_i \equiv f_i(\omega_i + \beta y)$, $i = 1, 2, \ldots, N$ are not conserved densities except for the trivial case when f_i are linear functions of their arguments.

To study the concrete exact solutions, we discussed some special examples for the ideal fluid without β term. It is not very difficult to verify that when $\beta = 0$, the MVIM possesses the following weak solutions ($r_i^2 \equiv (x - x_i(t))^2 + (y - y_i(t))^2$, $\theta_i \equiv \arctan[(y - y_i)/(x - x_i)]$)

$$\psi_i = a_i \ln r_i, \quad i = 1, \ldots, k, \tag{14}$$

$$= a_i(\ln^2 r_i - \theta_i^2), \quad i = k+1, \ldots, n, \tag{15}$$

$$= a_i r_i^{-\delta_i}\sin[\delta_i\theta_i + \theta_{i0}] \qquad i = n+1, \ldots, j, \tag{16}$$

$$= a_i e^{-\delta_i\theta_i}\sin[\delta_i \ln r_i + \theta_{i0}], \quad i = j+1, \ldots, N, \tag{17}$$

with arbitrary positive integers k, m, n, j, N, arbitrary constants a_i, δ_i, θ_{i0} and arbitrary functions $x_i \equiv x_i(t)$, $y_i \equiv y_i(t)$, $i = 1, \ldots, N$ and the corresponding vorticity sources are

$$\omega_i = \Delta\psi_i = -2a_i\pi\omega_{i0}\delta(x_i)\delta(y_i), \tag{18}$$

with $x_i \equiv x - x_{i0}$, $y_i \equiv y - y_{i0}$, $\omega_{i0} = 1$, $2\ln(r_i)$, $-\delta_i r^{-\delta_i}\sin(\delta_i\theta_i + \theta_{i0})$, and $\delta_i e^{-\delta_i\theta_i}\cos(\delta_i \ln r_i + \theta_{i0})$, respectively and $\delta(x_i)$ being the usual Dirac delta function.

It should be mentioned that all the solutions for the ψ_i listed in Eqs. (14)–(17) are weak source solutions. A weak solution of the model (4)–(5) means that it is an exact solution at all analytical areas. However, at the singular source points $\{x_i, y_i\}$, Eqs. (14)–(17) will not result exact zero identities but the zero distributions.

Point vortex source. The solution (14) is related to the multiple vortex sources. Figure 1 displays the structure of such type of single vortex source with the selection

$$\psi_1 = \ln r_1, \quad x_1 = y_1 = 0. \tag{19}$$

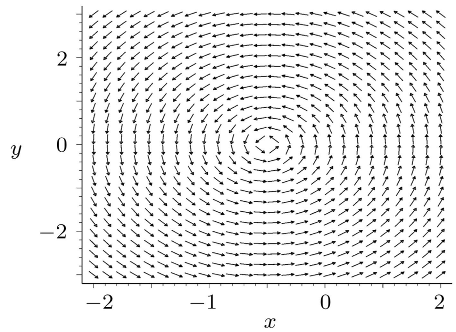

Fig. 1 The velocity field plot of the single vortex source with the stream function (20).

It should be mentioned/emphasized that to see the stream lines clearly, all the arrows in the figures in this Letter have the same length.

Vortex dipole source. The solution (15) is related to the multiple vortex dipole source solution. Figure 2 exhibits the single vortex dipole structure with the stream function

$$\psi_1 = \ln^2 r_1 - \theta_1^2\,, \quad x_1 = y_1 = 0\,. \tag{20}$$

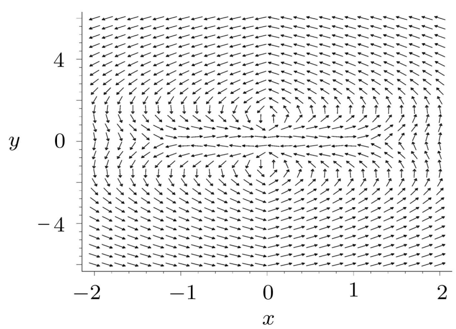

Fig. 2 The single vortex dipole structure of the EE with the stream function solution (20).

Multi-pole sources. The solution (16) is related to the multiple vortex-like multi-pole source solution. Figure 3(a) exhibits the single vortex quadrupole structure with the stream function

$$\psi_1 = r_1^{-2}\sin(2\theta_1) = \frac{2xy}{(x^2+y^2)^2}\,, \quad x_1 = y_1 = 0\,, \tag{21}$$

and Fig. 3(b) denotes the single vortex six-pole structure with

$$\psi_1 = r_1^{-3}\sin(3\theta_1) = \frac{y(3x^2-y^2)}{(x^2+y^2)^3}\,, \quad x_1 = y_1 = 0\,. \tag{22}$$

Cyclon sources with fractal structures. The solution (17) displays the structure of the multiple cyclon solution. We define a negative/positive cyclon as a vortex, which possesses a hole-like/source-like cycle. In other words, all the flows of the negative cyclon flow into the cycle while all those of the positive cyclon are flow out from the cycle. Figure 4 displays the special cyclon structures with limit cycles and the corresponding stream functions have the forms

$$\psi_{1\mp} = \exp(\mp 2\theta(t))\sin(\ln(x^2+y^2))\,, \quad x_1 = y_1 = 0\,, \tag{23}$$

where the up negative sign "−" is related to the negative cyclon, and the lower positive sign "+" corresponding to the positive cyclon.

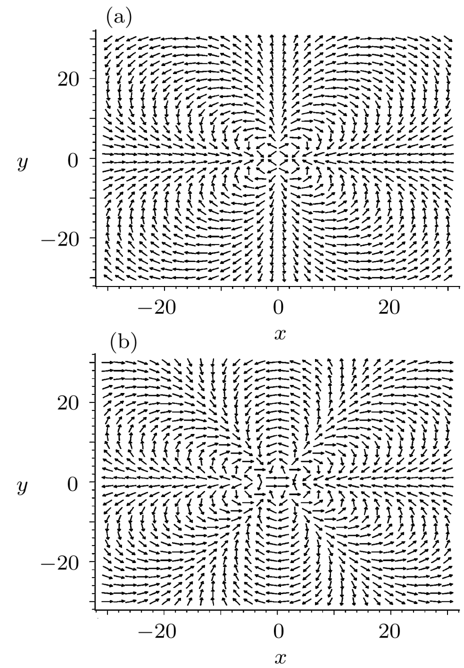

Fig. 3 The structures of the quadru-pole and six-pole with the stream functions (21) and (22) respectively.

It is interesting to mention that axes and tick-marks are removed in Fig. 4 because this kind of solution possesses a fractal structure. Concretely, if we plot the velocity field with the stream function (23) at the space regions

$$x = [-\alpha^n, \alpha^n]\,, \quad y = [-\alpha^n, \alpha^n], \quad \alpha \equiv e^{\pi}\,, \tag{24}$$

we can find exact same figures as shown in Fig. 4 for arbitrary integer $n > 1$.

Cyclon dipole sources with fractal structures. Positive and negative cyclons can be combined to produce many kinds of dipole sources. Figure 5 exhibits a special cyclon dipole structure with the stream function $\psi_1 = \psi_{1+} + \psi_{1-}$.

Obviously, similar to Fig. 4, one can find exactly same structures for the cyclon dipole sources when plotting the velocity field related stream function $\psi_{1+} + \psi_{1-}$ in the regions shown in Eq. (24)

In summary, a multiple vortex elastic interaction model (4)–(5) is proposed for an inviscid nondissipative and equivalent barotropic vorticity system in a beta-plane channel. It is found that the model possesses abundant symmetries and conservation laws. Especially, the self energy of every vortex and the interaction energy of every two vortices are conserved. However, the higher order vorticity moment density ω_i^k, $\forall k > 1$ for every vortex is not conserved even for $\beta = 0$, though the formally conservation law exists with an arbitrary density $f(\omega_{\rm eff})$ that is an arbitrary function of the total equivalent vorticity $\omega_{\rm eff} \equiv \sum_{i=1}^{N} \omega_i + \beta y$.

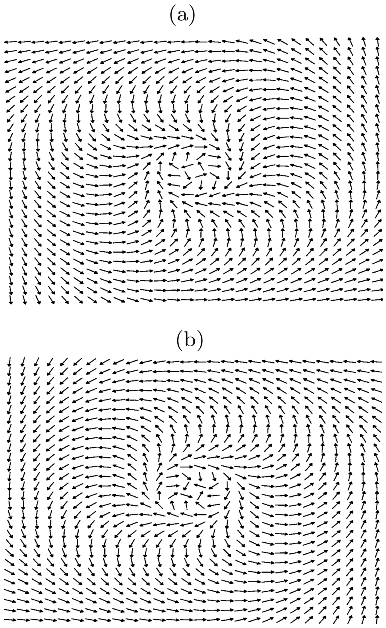

Fig. 4 (a) The structure of the special negative cyclon for ψ_{1-} given in Eq. (23); (b) The structure of the special positive cyclon for ψ_{1+}.

Some types of nontrivial exact weak solutions are obtained for the $\beta = 0$ case. In detail, multiple point vortex source solutions, multiple vortex dipole sources, multiple vortex multi-pole source solutions, multiple fractal cyclons and multiple fractal cyclon dipoles are obtained when we simply consider the zero vorticities $\omega_i = 0$. It seems to us that the multi-pole source solutions, fractal cyclon source solutions and fractal cyclon dipole sources have not yet been found before. The cyclon (not cyclone) is defined as a flow possesses a limit cycle where all flows flow into or out of the cycle. The fractal cyclon is named if a cyclon has a self-similar structure. In real nature, there exist various kinds of vortices, therefore, we hope that these kinds of vortex solutions might be observed from the real natural phenomena, or alternatively, from fluid experiments and other physical fields.

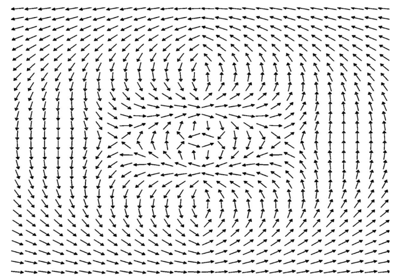

Fig. 5 The structure of a special cyclon dipole with the stream function is given by $\psi_1 = \psi_{1+} + \psi_{1-}$.

Though the multiple vortex interaction model is established, some kinds of conservation laws have been found and some special exact source solutions are obtained, various important problems are still open. For instance, is there any useful Lax pairs for the MVIM? How to find exact vortex source solutions of the model for $\beta \neq 0$? How to explore exact multiple analytical non-source solutions both for $\beta = 0$ and $\beta \neq 0$? How to discover any other types of exact analytical multiple vortex solutions with non-constant/non-zero vorticities? How to modify the MVIM (4)–(5) further to study other possible structures and interactions of the multiple vortices? Especially, to apply the model to real vortex physics, the non-elastic interactions must be involved. Which kinds of vortex interactions can be introduced in the non-elastic interaction models? All these important problems will be studied in near future.

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