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Equivalent and Nonequivalent Barotropic Modes for Rotating Stratified Flows*

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Abstract All the possible equivalent barotropic (EB) laminar solutions are firstly explored, and all the possible non-EB elliptic circulations and hyperbolic laminar modes of rotating stratified fluids are discovered in this paper. The EB circulations (including the vortex streets and hurricane like vortices) possess rich structures, because either the arbitrary solutions of arbitrary nonlinear Poisson equations can be used or an arbitrary two-dimensional stream function is revealed which may be broadly applied in atmospheric and oceanic dynamics, plasma physics, astrophysics and so on. The discovery of the non-EB modes disproves a known conjecture.

Keywords Rotating stratified flows, Equivalent barotropic modes, Nonequivalent barotropic modes
2000 MR Subject Classification 35Q35

1 Introduction

It is known that both the planetary rotations and stable vertical density stratification are important for the fluid motions. The flows in the rotating stratified fluids exhibit rich phenomena especially on the circulation vortices (see [1–2]) like the Great Jupiter's Red Spot, Dipolar vortex structures in (non-rotating) stratified fluids (see [3–4]), vortex street in soap films (see [5–6]), tripolar structure in homogeneous rotating fluid (see [7]), vortex dipoles in electrically forced magnetohydrodynamic flows (see [8]), tropospheric cyclones, hurricanes (see [9–10]), tornados, stratospheric polar vortices, oceanic Gulf Stream rings and atmospheric blockings (see [11–14]). The studies on the rotating stratified fluid systems are useful not only in fluid dynamics but also in many other physical fields including the condense matter, plasma physics,

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nuclear physics, astrophysics and cosmology (see [15–22]). Moreover, coherent vortex structures obtained from plasma experiments are analogous to the dynamics of pure-electron plasmas (see [23–25]).

By introducing buoyancy and advective time scales to a Boussinesq fluid with small Froude number, Lilly [26] split the velocity field into the propagating internal gravity waves (horizontally divergent and possessing no potential vorticity) and rotating stratified turbulence (horizontally nondivergent and possessing potential vorticity). The model equations of the latter have the form (see [26])

$$\begin{aligned} &u_t + uu_x + vu_y - fv = -p_x,\\ &v_t + uv_x + vv_y + fu = -p_y,\\ &p_z = -\rho,\\ &u_x + v_y = 0,\\ &\rho_t + u\rho_x + v\rho_y + w\rho_z = 0, \end{aligned} \tag{1.1}$$

where f is the Coriolis parameter, p is the pressure perturbation divided by a mean density ρ_0, ρ is the density perturbation scaled by $\frac{\rho_0}{g}$, u and v are horizontal velocities, and w is the vertical velocity w.

For a rotating homogeneous fluid with small Rossby number, the Taylor-Proudman theorem requires the velocity field to be vertically invariant, which gives vanishing vertical velocity under rigid boundary conditions. Therefore, Sun [27] assumes that an unforced rotating stratified turbulence decays toward a steady laminar state in absence of transient internal waves and vertical motions, that is,

$$\vec{u} = u\vec{i} + v\vec{j} + 0\vec{k}.$$

Such flows with vertical-varied horizontal velocities are also called pseudoplane motions (see [28]), for which system (1.1) becomes

$$\begin{aligned} &uu_x + vu_y - fv = -p_x,\\ &uv_x + vv_y + fu = -p_y,\\ &p_z = -\rho,\\ &u_x + v_y = 0,\\ &u\rho_x + v\rho_y = 0. \end{aligned} \tag{1.2}$$

The geostrophic empirical mode columnar structure is interpreted by the geostrophic version of (1.2) (see [29]). System (1.2) collectively forms a nonlinear model for baroclinic laminar flows that is steady, nondiffusive, and quasi-horizontal.

In Section 2, we review the known exact solutions and the conjecture made by Sun [27]. All the possible EB modes are discussed in Section 3. In Section 4, we discuss some non-EB modes. The last section is a short summary and discussion.

2 Known Exact Solutions and a Conjecture

Before finding possible exact solutions to (1.2), the definitions used in [27] should be rewritten down here.

Definitions 2.1 (see [27]) *The fluid is barotropic if density is a function of pressure only, that is, isobaric surfaces and isopycnal surfaces coincide; otherwise, the fluid is baroclinic. A baroclinic flow is EB if the streamlines on each plane align vertically or, equivalently, if the horizontal velocity vector does not change direction vertically. More clearly, the fluid is barotropic iff the pressure of* (1.2) *is a function of* $z+h$*, i.e.,*

$$p = p(z+h) \tag{2.1}$$

with an arbitrary function $h \equiv h(x,y)$ *while the fluid is called EB of* (1.2) *iff*

$$\psi_x = F\psi_y \tag{2.2}$$

for arbitrary $F \equiv F(x,y)$*.*

To derive the model (1.2), Sun hypothesized that the formation mechanism for coherent structures in rotating stratified flows is fundamentally baroclinic. However, to find exact baroclinic solutions in fluid dynamics is very difficult, and there is little progress in this direction. In [27], Sun found five types of exact solutions:

(i) barotropic tilting vortex solution

$$u = fy, \quad v = -f(x+z), \quad p = F(z), \quad \rho = -F'(z);$$

(ii) degenerate EB elliptical solution

$$u = -2by, \quad v = 2ax, \quad p = 2ab(x^2+y^2) + f(ax^2+by^2) + F(z), \quad \rho = -F'(z);$$

(iii) EB zonal jet solution

$$u = z\sin y, \quad v = 0, \quad p = fz\cos y + F(z), \quad \rho = -f\cos y - F'(z);$$

(iv) irrotational EB circular solution

$$u = -z\frac{y}{r^2},$$

where $r = \sqrt{x^2+y^2}$, $v = z\frac{x}{r^2}$, $p = fz\ln r - \frac{z^2}{2r^2}$, $\rho = \frac{z}{r^2} - f\ln r$;

(v) rotational EB circular solution

$$u = -2yz, \quad v = 2xz, \quad p = 2z^2r^2 + fzr^2, \quad \rho = -(f+4z)r^2.$$

Solution (i) is a free inertial oscillation in the absence of horizontal pressure gradients, with one-half of a pendulum day as the period. Solution (ii) means the existence of a nonaxisymmetric vortex in a degenerate baroclinic regime. Solution (iii) is related to a unidirectional jet in geostrophic balance. The combination of two EB circular solutions (iv) and (v) may yield physically realistic models such as the baroclinic Rankine vortex for meddies. Except the degenerate ones, the only baroclinic solutions obtained by Sun are the EB circular vortex and unidirectional jet. Based on the fact that all the known solutions are either barotropic or EB, a conjecture is proposed in [27].

Conjecture 2.1 (see [27]) *Baroclinic solutions to* (1.2) *are always EB.*

Now, the two important questions are: How to find possible baroclinic modes of (1.2)? Is the conjecture correct?

3 EB Modes

Here, we try to find all the possible EB modes of the baroclinic laminar model (1.2). From the incompressible condition

$$u_x + v_y = 0,$$

we can introduce the stream function ψ as

$$u = -\psi_y, \quad v = \psi_x. \tag{3.1}$$

After introducing the stream function as in (3.1), five equations shown in (1.2) are reduced to two equations for the single function ψ

$$J(\psi, K_z) - (\zeta + f)J(\psi, \psi_z) = 0, \tag{3.2}$$

$$J(\psi, \zeta) = 0, \tag{3.3}$$

where

$$K \equiv \frac{1}{2}\psi_x^2 + \frac{1}{2}\psi_y^2, \quad \zeta \equiv \psi_{xx} + \psi_{yy}, \quad J(a, b) \equiv a_x b_y - a_y b_x.$$

(3.2) is just the last equation of (1.2) while (3.3) is the consistent condition of the first two equations of (1.2), i.e., $p_{xy} - p_{yx} = 0$. Once the stream function ψ is solved out from (3.2) and (3.3), the velocity components are obtained immediately from (3.1), and the pressure can be solved out from the consistent equations, i.e., the first two equations of (1.2), while the density is only a simple differentiation of the pressure with respect to z.

From (2.2) and (3.2), it is easy to find the only two possible cases of EB.

Case 1

$$\psi_{yz} = 0; \tag{3.4}$$

Case 2

$$F_y + FF_x = 0. \tag{3.5}$$

Case 1 EB modes with an arbitrary nonlinear Poisson flow. For the $\psi_{yz} = 0$ case, we have the stream function

$$\psi = \phi(x, y) + \psi_0(z) \tag{3.6}$$

with $\psi_0(z)$ being an arbitrary function of z while $\phi \equiv \phi(x, y)$ is a solution to an arbitrary nonlinear Poisson equation

$$\phi_{xx} + \phi_{yy} = g(\phi), \tag{3.7}$$

where $g(\phi)$ is an arbitrary function of ϕ. Once the Poisson equation (3.7) is solved, the other quantities can easily be found. The results read

$$u = -\phi_y, \quad v = \phi_x, \tag{3.8}$$

$$p = \frac{1}{2}\phi_y^2 + \int \phi_x \phi_{yy} \mathrm{dx} + f\phi + \phi_0(y) + p_0(z), \tag{3.9}$$

where $p_0(z)$ is an arbitrary function of z while $\phi_0(y)$ should be appropriately fixed such that the first two equations of (1.2) are compatible.

In Figure 1, a special vortex street solution $((m, n) \equiv (a\frac{c}{b}, a\frac{b}{c}))$,

$$\psi = \phi = 4 \arctan\left(a\mathrm{sn}(bx, m)\mathrm{sn}(cy, n)\right), \tag{3.10}$$

where $a,\ b$ and c are constants and $\mathrm{sn}(bx, m)$ is the standard Jacobian elliptic function with modula m, is shown with the parameter selections $a = \frac{1}{8},\ b = 2,\ c = \frac{1}{2}$.

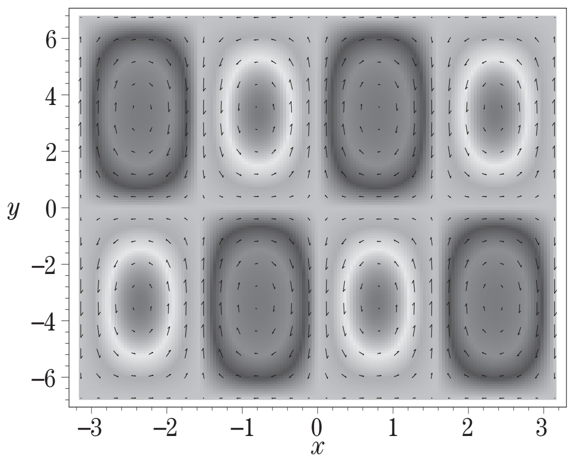

Figure 1 The density plot of the vortex street solution (3.4) and the vector field plot of the corresponding velocity field expressed by (3.12) and (3.13) with the parameter selections $a = \frac{1}{8},\ b = 2,\ c = \frac{1}{2}$.

Corresponding to the solution (3.10), the arbitrary function of Poisson equation is fixed as

$$g(\phi) = -(b^2 + c^2)(1 + a^2)\sin(\phi), \tag{3.11}$$

and the other physical quantities are

$$u = -\frac{4ac\mathrm{sn}(bx,m)\mathrm{cn}(cy,n)\mathrm{dn}(cy,n)}{1 + a^2\mathrm{sn}^2(bx,m)\mathrm{sn}^2(cy,n)}, \tag{3.12}$$

$$v = \frac{4ab\mathrm{cn}(bx,m)\mathrm{dn}(bx,m)\mathrm{sn}(cy,n)}{1 + a^2\mathrm{sn}^2(bx,m)\mathrm{sn}^2(cy,n)}, \tag{3.13}$$

$$p = f\phi - 8b^2\frac{c^2\mathrm{sn}^2(bx,m) + b^2\mathrm{sn}^2(cy,n)}{1 + a^2\mathrm{sn}^2(bx,m)\mathrm{sn}^2(cy,n)} + g, \tag{3.14}$$

where $g \equiv g(z)$ is an arbitrary function of z.

We should remark that there are many exact solutions to the nonlinear Poisson equation which is widely used to explain the two-dimensional guiding-center plasma or two-dimensional line vortex system in fluids. For example, the exact vortex solutions to sinh-Poisson equation describe statistical equilibrium states of the two-dimensional guiding-center plasma (see [30–31]). Exact solutions to sinh-Poisson equation for doubly periodic domains or doubly periodic vortex arrays in the plane explain a stream function configuration of a stationary two-dimensional (2D) Euler flow (see [32–34]). However, in this paper, an arbitrary Poisson equation is derived, which may have a profound physical significance in the theory of vortex dynamics such as in ideal incompressible (Euler) fluid, plasma in strong magnetic field, planetary atmosphere, non-neutral plasmas, etc.

Cases 2 EB symmetric circulations. For the $F_y + FF_x = 0$ case, it is straightforward to prove that the only possible modes are

$$\psi = \psi_0(r,z), \quad r \equiv c_1(x^2 + y^2) + c_2x + c_3y, \tag{3.15}$$

$$u = -\psi_{0r}(2c_1y + c_3), \quad v = \psi_{0r}(2c_1x + c_2), \tag{3.16}$$

$$p = 2c_1\int \psi_{0r}^2 \mathrm{d}r + f\psi_0 + p_0(z), \tag{3.17}$$

where c_1, c_2 and c_3 are arbitrary constants while $p_0 \equiv p_0(z)$ and $\psi_0 \equiv \psi_0(r,z)$ are arbitrary functions of the indicated variables.

It is clear that there exist abundant symmetric circulation modes ($c_1 \neq 0$ in (3.15)) and jet modes ($c_1 = 0$ in (3.15)) because the stream function is an arbitrary function of two variables r and z. The richness of the symmetric circulations for the rotational fluids is natural as one observed in both the ocean and the atmosphere. Actually, both in the atmosphere and in the ocean, there are also many kinds of nonsymmetric circulations.

In Figure 2(a), a special second type of baroclinic symmetric EB mode is plotted for the velocity field ($r \equiv x^2 + y^2 - 2$)

$$u = 2(z-1)y\mathrm{sech}[(1-z)r]\mathrm{sech}(1-z), \tag{3.18}$$

$$v = 2(1-z)x\mathrm{sech}[(1-z)r]\mathrm{sech}(1-z), \tag{3.19}$$

which is related to the stream function solution (3.15)

$$\psi = \mathrm{sech}(1-z)\arctan\{\sinh[(1-z)r]\}\,. \tag{3.20}$$

Correspondingly, the pressure has the form

$$\begin{aligned} p =\;& 2(1-z)\tanh[(1-z)r]\mathrm{sech}^2[(z-1)r] \\ & - 2f\mathrm{sech}(1-z)\arctan\{\exp[(z-1)r]\} + p_0, \end{aligned} \tag{3.21}$$

where p_0 is still an arbitrary function of z.

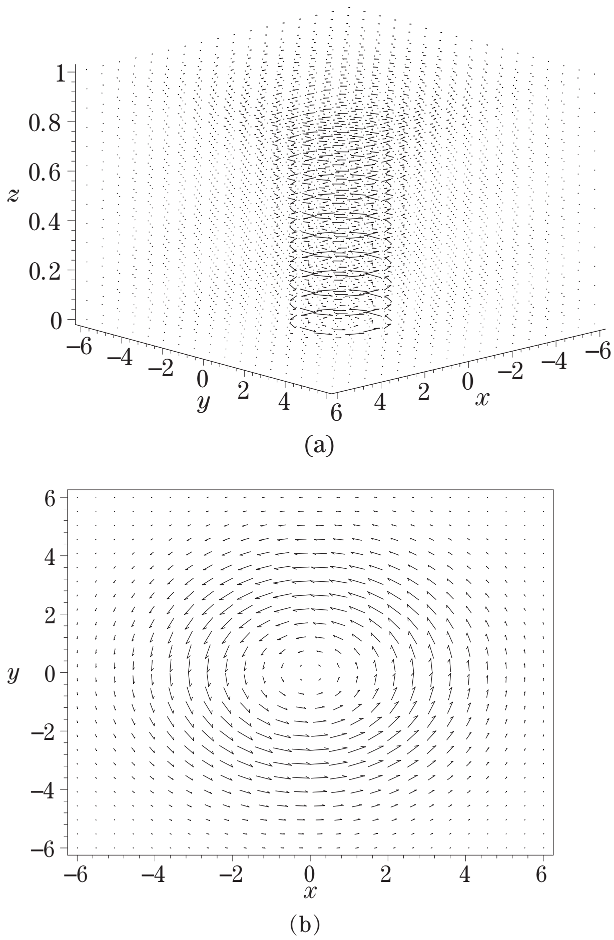

Figure 2 (a) A 3-dimensional vortex solution described by the vector velocity field (3.18)–(3.19). (b) The hurricane like structure which is the bird's eye view of (a).

Overlooking Figure 2(a), we get Figure 2(b) which exhibits a hurricane-like circulation form with a hurricane eye.

4 Non-EB Elliptic and Hyperbolic Modes

To find a nonsymmetric circulation, we restrict ourselves to finding elliptic or hyperbolic modes of (1.2). An elliptic or hyperbolic mode is defined as its stream lines are elliptic and/or hyperbolic curves. In other words, the stream function ψ has the form

$$\psi = \psi(a_1(x-x_0)^2 + a_2(y-y_0)^2, z) \tag{4.1}$$

with $a_1a_2 > 0$ for elliptic, $a_1a_2 < 0$ for hyperbolic modes and where $a_1 \equiv a_1(z)$, $a_2 \equiv a_2(z)$, $x_0 \equiv x_0(z)$, $y_0 \equiv y_0(z)$ and $\psi_0 \equiv \psi_0(z)$ being arbitrary functions of z.

Substituting (4.1) into (3.3), one can easily find

$$\psi_\xi \psi_{\xi\xi}[a_2 - a_1][y-y_0][x-x_0] = 0, \tag{4.2}$$

where

$$\xi \equiv a_1(x-x_0)^2 + a_2(y-y_0)^2.$$

From (4.2), we know that the only case is $\psi_{\xi\xi} = 0$ for nonsymmetric ($a_2 \neq a_1$) modes, i.e.,

$$\psi = a_1(x-x_0)^2 + a_2(y-y_0)^2 + \psi_0. \tag{4.3}$$

Requiring (4.3) to be an exact solution, one has to put some further constraints on the functions a_1, a_2, x_0 and y_0 because the stream function must satisfy not only (3.3) but also (3.2).

Substituting (4.3) into (3.2), we have

$$\begin{aligned} &(y-y_0)(x-x_0)[a_1a_{2z}(f+2a_1-2a_2) - a_2a_{1z}(f+2a_2-2a_1)] \\ &- a_1a_2[(f+2a_1)(x-x_0)y_{0z} - (f+2a_2)(y-y_0)x_{0z}] = 0. \end{aligned} \tag{4.4}$$

From (4.4), it is not difficult to see that except the trivial EB solution with a_1, a_2, x_0 and y_0 being all z independent constants, there are three nontrivial non-EB solutions.

Case 1 Elliptic or hyperbolic non-EB modes with rotational shape as the height z changes

$$\psi_\pm = \pm\frac{1}{2h} g_\pm \eta_\pm^2 + \psi_0, \tag{4.5}$$

$$u_\pm = \frac{1}{h} g_\pm (y-y_0),\ v = g_\pm h(x-x_0), \tag{4.6}$$

$$p_\pm = p_0 + \frac{1}{2} g_\pm \Big(g_\pm \eta^2 \pm f \frac{\eta_\pm^2}{h} \Big), \tag{4.7}$$

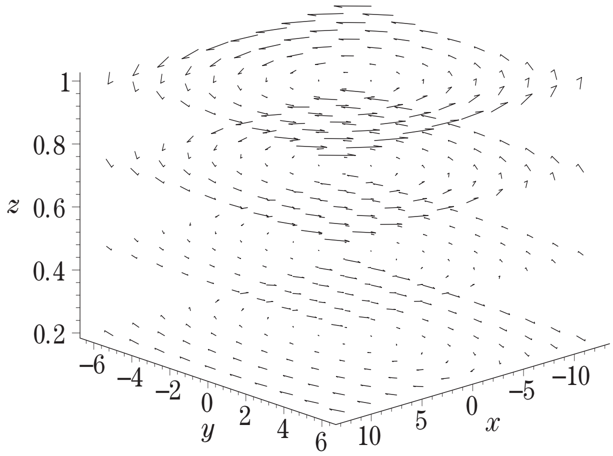

Figure 3 Baroclinic elliptic circulation for the vector velocity field described by (4.8)–(4.9).

where h, ψ_0 and p_0 are arbitrary functions of z, $\{x_0, y_0, c_1\}$ are arbitrary constants while

$$\begin{aligned}\eta_\pm^2 &\equiv (y-y_0)^2 \pm h^2(x-x_0)^2,\\ \eta^2 &\equiv (x-x_0)^2+(y-y_0)^2,\\ g_+ &\equiv c_1 - \operatorname{arctanh}(h),\\ g_- &\equiv c_1 - \arctan(h).\end{aligned}$$

The upper sign is related to the baroclinic elliptic circulation while the lower sign corresponds to the baroclinic hyperbolic wave case.

Figure 3 displays a special structure for an elliptic circulation with the velocity field

$$u = \frac{y}{10}(1+z^2)\Big(\operatorname{arctanh}\frac{1}{1+z^2} - \frac{3}{2}\Big), \tag{4.8}$$

$$v = \frac{x}{10(1+z^2)}\Big(\frac{3}{2} - \operatorname{arctanh}\frac{1}{1+z^2}\Big), \tag{4.9}$$

which corresponds to the selections

$$h = \frac{1}{1+z^2}, \quad c_1 = \frac{3}{2}, \quad x_0 = y_0 = 0, \quad f = \frac{1}{10}$$

in (4.6).

From (4.5)–(4.7), we find that the elliptic circulation possesses some interesting properties:

(i) The circulation center is independent of the height z.

(ii) The length of the elliptic axes can be changed to z and then the circulation shape is rotated as z changes.

(iii) All the quantities, such as the stream function, the velocity field and the pressure and density, possess elliptic distributions.

(iv) The rotation direction of the vortex may be changeable if $g_+ = c_1 - \operatorname{arctanh}(h) = 0$ has a solution. Otherwise, the rotation direction of the vortex will be independent of the height variable z.

Case 2 Baroclinic elliptic or hyperbolic non-EB mode with skew center.

$$\psi = c(x - x_0(z))^2 - \frac{f}{2}(y - y_0)^2 + \psi_0(z), \tag{4.10}$$

$$u = f(y - y_0), \quad v = 2c(x - x_0(z)), \tag{4.11}$$

$$p = p_0(z) + \frac{1}{2}f(2c + f)(y - y_0)^2, \tag{4.12}$$

where $x_0(z)$, $\psi_0(z)$ and $p_0(z)$ are arbitrary functions of z and y_0 and c are arbitrary constants. When $c < 0$, the solution (4.10)–(4.12) is related to the baroclinic elliptic non-EB circulation while the baroclinic hyperbolic non-EB mode is governed by $c > 0$.

Case 3 Second kind of Baroclinic elliptic or hyperbolic non-EB mode with skew center.

$$\psi = c(y - y_0(z))^2 - \frac{f}{2}(x - x_0)^2 + \psi_0(z), \tag{4.13}$$

$$v = -f(x - x_0), \quad u = -2c(y - y_0(z)), \tag{4.14}$$

$$p = p_0(z) + \frac{1}{2}f(2c + f)(x - x_0)^2, \tag{4.15}$$

where $y_0(z)$, $\psi_0(z)$ and $p_0(z)$ are arbitrary functions of z, and x_0 and c are arbitrary constants. This case is the same as that obtained by making the transformation $\{x, x_0, u\} \leftrightarrow \{y, y_0, -v\}$ in Case 2.

Different from the first type of baroclinic non-EB modes shown by (4.5)–(4.7), $\{x_0, y_0\}$, the center of the Cases 2 and 3 of the baroclinic non-EB circulations, is changeable as the height changes, while the length of axes of the circulation is independent of z. That means the circulation has fixed shape with skew center. On the other hand, the pressure and the density distributions have no circulation structure, though the stream function and the velocity field do. The rotation direction of the vortex is always independent of the height variable.

5 Summary and Discussions

In summary, the fluid systems on the earth such as the oceans and atmosphere have to be studied under the rotational coordinates, and the laminar modes for the rotating stratified flows have not yet been studied well, though the related topics are successfully studied for the usual irrotational fluid systems.

In this paper, we have studied some types of steady equivalent and nonequivalent barotropic modes for rotating stratified flows for the Lilly-Sun model (see [26–27]). Usually, to find some exact baroclinic solutions for the rotating fluids is very difficult and only some quite special solutions are found. By using the proper definitions, all the possible EB models is obtained.

The first type of EB modes are determined up to an arbitrary symmetric Poisson equation, which allows us to get infinitely many exact solutions including vortex street like solutions. This may help us better understand of statistical and magnetohydrodynamic equilibrium states of the two-dimensional guiding-center plasma and the evolution and stability of periodic vortex arrays in two-dimensional Euler flows. The second type of EB solutions is quite free because of the existence of an arbitrary stream function with two arbitrary variables. This kind of solutions exhibits rich structures of the jet modes and the symmetric circulations including some tropical cyclone-like vortices structures.

In addition to the abundant symmetric circulations, all the possible (three types of) elliptic circulations and/or hyperbolic modes are found. It is interesting that the discovery of this kind of solutions disproves Sun's conjecture (see [27]) because they are baroclinic and non-EB modes. For the first type of nonsymmetric circulations, the length of the elliptic axes, and the rotation directions may be changed with respect to the height z, while their circulation center is independent of z. For the second and third types of nonsymmetric circulations, the length of the elliptic axes and the rotation directions are independent of z while the circulation center may be skewed through the change of z.

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